Entropic Repulsion for a Gaussian Membrane Model in the Critical and Supercritical Dimensions

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Zusammenfassung

In dieser Dissertation wird ein stochastisches Modell zufälliger Grenzflächen untersucht. Mathematische Modelle zufälliger Phasengrenzflächen in der statistischen Physik sind seit eingen Jahrzehnten ein Forschungsgebiet von erheblichem Interesse. Mathematisch werden solche Modelle durch eine Familie von Zufallsgrössen auf einem d-dimensionalen Gitter beschrieben, zusammen mit einer Energiefunktion (Hamiltonfunktion). Die Zufallsgrössen werden dabei als Höhenkonfiguration betrachtet. Das bedeutet, sie beschreiben die Höhe der Phasengrenzfläche in Bezug auf eine Refernzfläche. Via die Hamiltonfunktion wird ein Gibbsmass auf der Menge der Höhenkonfigurationen definiert.

Eine wichtige Klasse solcher zufälliger Grenzflächen sind die Gradientmodelle. Vereinfacht gesagt sind dies Modelle, für welche die energetisch günstigen Konfigurationen überall nahezu konstant sind, das heisst, die bevorzugten Konfigurationen in Gradientmodellen sind im wesentlichen flach. Die wichtigste Eigenschaft für den mathematischen Zugang zu diesen Gradientmodellen ist deren Irrfahrtendarstellung. Das bedeutet, dass viele wichtigen Grössen, wie beispiesweise der Erwartungswert, die Kovarianzen oder die Zustandssumme, durch Ausdrücke dargestellt werden können, welche von Irrfahrten auf Gittern herrühren. Diese Darstellung wird bei der Untersuchung von Gradientmodellen auf vielfältige Weise ausgenutzt.

Allerdings beschreiben Gradientenmodelle bei weitem nicht alle Eigenschaften physikalischer Grenzflächen. Eine grosse Klasse von beträchtlichem Interesse sind die sogeannten semiflexiblen Membrane, oder semiflexiblen Polymere. Hier sind die physikalisch günstigen Konfigurationen diejenigen, welche nicht nur die Höhendifferenzen minimieren, sondern auch möglichst konstante Krümmung besitzen. Die Hamiltonfunktion eines solchen Modells hängt deshalb nicht nur vom Gradienten, sondern auch von den zweiten Ableitungen der Höhenkonfiguration ab. Diese Abhängigkeit von den zweiten Ableitungen hat jedoch keine Irrfahrtendarstellung mehr. Deshalb können die meisten mathematischen Methoden welche für Gradientmodelle benutzt werden, nicht einfach so auf semiflexible Membrane übertragen werden.

In dieser Dissertation untersuchen wir ein Gauss'sches Grenzflächenmodell, bei dem die Hamiltonfunktion abhängig ist vom Laplace-Operator angewandt auf die Höhenkonfiguration, und welches deshalb keine Irr-

fahrtendarstellung besitzt. Wir interessieren uns für die Frage der entropischen Abstossung. Dieser Begriff bezieht sich auf das Verhalten der Grenzfläche in Gegenwart einer “festen Wand”, beziehungsweise auf die Einschränkung auf Flächen, welche auf einem bestimmten Gebiet des Gitters positiv sind. Die entropische Abstossung ist ein wichtiger Schritt zum Verständnis von Benetzungsübergängen. Für unser Modell ist es einfach zu sehen, dass in den Dimensionen grösser als fünf das Gibbsmass auf dem unendlichen Volumen existiert. In diesem superkritischen Regime können wir das Fehlen einer Irrfahrtendarstellung durch analytische Methoden und einer sorgältigen Anwendung von Normalverteilungsabschätzungen umgehen.Damit können wir das genaue asymptotische Verhalten der Wahrscheinlichkeit für das Ereignis, dass die Grenzfläche auf einem bestimmten Gebiet positiv ist, bestimmen. Mit diesem Resultat können wir dann beweisen, dass die Fläche durch eine Wand der Grössenordnung N auf eine Höhe der Ordnung $\sqrt{\log N}$ gedrängt wird, was bedeutet, dass das Modell tatsächlich eine entropische Abstossung zeigt.

In tieferen Dimensionen ist die Situation etwas komplizierter, da das Gibbsmass auf dem unendlichen Volumen nicht existiert. Es müssen Randbedingungen eingeführt werden um ein Gibbsmass zu erhalten, und diese führen zu Problemen in der Untersuchung der Kovarianzen. In dieser Arbeit entwickeln wir analytische Methoden, inspiriert von der Theorie über die Regularität von Randwertproblemen, um das asymptotische Verhalten der Varianzen auf endlichem Volumen zu untersuchen. Damit beweisen wir, dass die Kovarianzen, obwohl sie keine direkte Irrfahrtendarstellung besitzen, angenähert werden können durch eine Grösse, welche die Anzahl Überschneidungen zweier Irrfahrten beschreibt. Diese Resultate wenden wir dann auf die kritische Dimension $d = 4$ an. Wir können mit unseren Methoden die Korrelationen der Grenzfläche in einem Mass kontollieren, welches ausreicht, um eine bereits bekannte Multiskalen-Methode für ein Gauss’sches Gradientmodell anzuwenden. Damit beweisen wir ein Resultat über das Verhalten des Maximums, und geben das asymptotische Verhalten für die Wahrscheinlichkeit, dass die Grenzfläche positiv ist. Wir beweisen ausserdem eine untere Schranke der Ordnung $\log N$ für die Höhe der Grenzfläche bedingt auf Positivität auf einem Gebiet der Grössenordnung N.

Zusammenfassend zeigen wir in dieser Dissertation entropische Abstossung für ein Membranmodell in den kritischen und superkritischen Dimensionen, wobei wir neue – analytische und stochastische – Methoden entwickeln, um mit gewissen Modellen zufälliger Grenzflächen ohne Irrfahrtendarstellung umzugehen, nicht nur für das unendliche, sondern auch zu einem gewissen Grad auf endlichem Volumen.

Summary

In this thesis, we study a probabilistic model of random interfaces. Mathematical models for interfaces between different phases in statistical physics have been a topic of considerable interest in the last decades. Mathematically, such interfaces are described by a family of random variables, indexed by the d-dimensional integer lattice, which are considered a height configuration, that is, they indicate the height of the interface above a reference hyperplane. The model is defined in terms of an energy function (Hamiltonian), which defines a Gibbs measure on the set of height configurations.
An important class of such random interfaces are the "gradient models". Generally speaking, the energetically favourable configurations for such models are those with approximately constant height everywhere. The major tool in the mathematical approach on gradient models is their random walk representation, which means that many important quantities such as mean, covariances, or the partition function, can be expressed in terms of a random walk on the integer lattice. This representation is exploited in may ways in the investigation of the behaviour of gradient models.
However, gradient models do not capture all features of physical interfaces. A large class of considerable interest are the so-called semiflexible membranes or semiflexible polymers. Here, the physically favourable configurations are those which do not only minimize the height differences, but also have approximatively constant curvature. The Hamiltonian of such a model does therefore not only depend on the gradient of the height configuration, but also on the second derivatives. This dependence on second derivatives however does not have a random walk representation any more. Hence most methods of gradient models can not be applied to semiflexible membranes.
In this thesis, we consider a Gaussian model where the Hamiltonian depends on the Laplacian of the height configuration and which does therefore not have a random walk representation. We are interested in the question of entropic repulsion. This refers to the behaviour of the interface in the presence of a "hard wall", meaning a constraint on the interface to be positive on a certain region of the lattice. Entropic repulsion is an important step in studying the wetting transition for random interfaces. For the model we consider, it is easy to see that in dimen-

sions ≥ 5, the infinite volume Gibbs measure exists. In this supercritical regime, we can overcome the lack of a random walk representation by analytical methods and a somewhat more careful use of Gaussian estimates than necessary for the corresponding Gaussian gradient model. We can compute precise asymptotics of the probability that the interface is positive on a certain region. This results then enables us to show that the interface is pushed to height of order $\sqrt{\log N}$ by a "hard wall"of order N thus showing that our model does display the expected entropic repulsion behaviour.
The situation is somewhat more complicated in the lower dimensions where the infinite volume measure does not exist. The boundary conditions that need to be introduced in order to obtain a Gibbs measure cause difficulties in the analysis of the covariances. In this thesis, we develop analytical methods inspired by the theory of regularity for boundary value problems, in order to study the asymptotic behaviour of the finite volume variances. With these methods we show that the covariances – although they don't have a direct random walk representation – can be approximated by an expression involving intersections of random walks. We then apply these methods to the critical dimension $d = 4$. With our methods, we gain sufficient control on the correlations of the interface in order to apply the multiscale methods developed for the Gaussian gradient model. Using these, we prove a result on the behaviour of the maximum of the interface, and then give the asymptotics on the probability that the interface is positive. We also prove a lower bound of order $\log N$ on the height of the interface conditioned to be positive on a box of side-lenght N.
Summarizing, in this thesis we prove entropic repulsion behaviour for a membrane model in the critical and supercritical dimensions, developing new methods – analytic and probabilistic – to deal with certain interface models without a random walk representation, not only in the infinite volume case, but to some extent also in the finite volume.

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Chapter 1
The model and main results

1.1 Random interface models

The theory of random interface models belongs to the field of statistical physics. Many models traditionally studied in statistical physics display interface properties. As an example, let us consider the classical 2-dimensional Ising model on a box of size $N \times N$, where we impose plus-boundary conditions on one side, and minus-boundary conditions on the other three. At low temperature, with high probability we will see a configuration consisting of two phases, a plus-phase near the plus-boundary, and a minus-phase. The region separating the two phases is what we call an interface. On the microscopical scale, it is not necessarily sharply defined. However, one would like to study the macroscopical properties of such an interface, such as for example the height above a reference hyperplane (in our example the $x-$axis), its curvature, flucuations, behaviour under constraints etc. These macroscopical properties are assumed to be in some sense universal, not particular characteristics of the underlying model, in our example the Ising model. This is a motivation to introduce a class of probabilistic models, which describe the interface itself as a microscopic object, and not as a property of some other model.
In this section, we introduce in some generality a class of continuous effecive interface models, so-called *gradient models*. It should be noted that the *membrane model*, which is the model considered in this thesis, does *not* belong to this class of gradient models. However, it is helpful for the physical understanding to compare it to the *lattice Gaussian free field*, which is probably the best understood example of such a continuous effective interface model. Also, the basic setting is the same: Let

$$\varphi = \{\varphi_x\}_{x\in\mathbb{Z}^d}$$

be a family of real-valued random variables indexed by the $d-$dimensional integer lattice. Such a family can be interpreted as a $d-$dimensional interface in $d+1-$dimensional Euclidean space $\mathbb{R}^{d+1}$ in the following manner: We think of the φ_x as *height variables*, indicating the height of the interface above the (integer valued) point x in the $d-$dimensional reference hyperplane. We obtain a $d-$dimensional surface in $\mathbb{R}^{d+1}$ by interpolating the heights linearly between the integer points.
We will in general forget about the interpolation, and call any configuration $\{\varphi_x\}_{x\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$ an *interface*. We identify the family $\{\varphi_x\}_{x\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$ with the mapping

$$\varphi : \mathbb{Z}^d \to \mathbb{R} : \varphi(x) = \varphi_x.$$

For notational convenience, we will switch between these two viewpoints, but use the notation φ_x in both cases.
We now want to introduce a probability measure on the set of interface configurations. Let $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ be endowed with the product topology. Let Λ be a finite subset of $\mathbb{Z}^d$. We fix a configuration $\{\psi_x\}_{x\in\mathbb{Z}^d\setminus\Lambda}$ which plays the role of a boundary condition. To each configuration φ we associate an *energy* given by a *Hamiltonian* $H_\Lambda^\psi(\varphi)$. Then a probability measure on Ω is given (formally) by

$$P_\Lambda^{\psi,\beta}(d\varphi) = \frac{1}{Z_\Lambda^{\psi,\beta}} \exp\left(-\beta H_\Lambda^\psi(\varphi)\right) \prod_{x\in\Lambda} d\varphi_x \prod_{x\notin\Lambda} \delta_{\psi_x}(d\varphi_x). \tag{1.1}$$

Here, $\beta \geq 0$ is an additional parameter (the inverse temperature), $d\varphi_x$ (for each $x \in \Lambda$) is the one-dimensional Lebesgue-measure, δ_{ψ_x} the Dirac mass at ψ_x and $Z_\Lambda^{\psi,\beta}$ the constant which normalises $P_\Lambda^{\beta,\psi}$ to a probability measure (if it is finite). In other words, if $P_\Lambda^{\beta,\psi}$ exists, it is the probability measure on the set of configurations restricted to be equal to ψ outside Λ, which has density $\exp(-\beta H_\Lambda^\psi)$ with respect to the product Lebesgue measure on $\mathbb{R}^\Lambda$. This is the general framework in which we want to study random interfaces. Of course, while we haven't fixed the Hamiltonian, this does not yet make much sense. So let us turn now to more concrete models.
A *gradient model* (or $\nabla-$*model*) is a random interface model in the context we just described, where the Hamiltonian is given by

$$H_\Lambda^\psi(\varphi) = \frac{1}{2} \sum_{x,y\in\Lambda} p_{x,y} V(\varphi_x - \varphi_y) + \sum_{x\in\Lambda, y\notin\Lambda} p_{x,y} V(\varphi_x - \varphi_y). \tag{1.2}$$

Here, $V : \mathbb{R} \to \mathbb{R}$ is an even convex function with $V(0) = 0$, the *potential*, and $p_{x,y}$ is the transition matrix of a random walk on the lattice $\mathbb{Z}^d$ which we assume here to have finite range (there are more general conditions under which the measure is well defined). In this case, (1.1) defines a probability measure on $\mathbb{R}^\Lambda$.

There is much literature available on this class of random interface models, for an overwiew see for example the lecture notes by Funaki [10], Giacomin [12] or Velenik [22]. An important special case is the one where $V(x) = x^2$. In this case, the Hamiltonian can be rewritten in the following form:

$$H_\Lambda^\psi(\varphi) = \frac{1}{2} \sum_{x,y\in\Lambda} \varphi_x (I-P)_\Lambda(x,y)\varphi_y + \sum_{x\in\Lambda} m_x^\psi \varphi_x, \tag{1.3}$$

where $(I - P)_\Lambda = (\delta(x,y) - p_{x,y})_{x,y\in\Lambda}$, and the m_x^ψ are real-valued coefficients. This shows that the field $\{\varphi_x\}_{x\in\Lambda}$ is Gaussian, and its covariance matrix is given by the Green's function of the random walk $(X_n)_{n\geq 0}$ with transition matrix $p_{x,y}$ killed at the exit of Λ, that is,

$$\mathrm{cov}_\Lambda(\varphi_x, \varphi_y) = (I-P)_\Lambda^{-1}(x,y) = \mathbb{E}^x\left(\sum_{n=0}^{\tau_\Lambda - 1} 1_{\{X_n = y\}}\right), \tag{1.4}$$

where $\tau_\Lambda = \inf\{n \geq 0 : X_n \notin \Lambda\}$. This is called the *random walk representation* of the covariances. There is a random walk representation for the mean as well, but in this thesis, the expression for the covariances will be of particular importance.

Since a Gaussian distribution is uniquely determined by its mean and covariances, it is of fundamental interest to find expressions for these quantities, especially if one can use these expressions to get information on the qualitative and quantitative behaviour of the interface model. The best quantitative estimates on the Green's function and related quantities we get if X_n is the nearest-neighbour simple random walk on $\mathbb{Z}^d$, that is, $p_{x,y} = (2d)^{-1}1_{\{|x-y|=1\}}$. This random interface model is called *lattice Gaussian free field* or *harmonic crystal*. Its (formal) Hamiltonian is given by

$$H(\varphi) = \frac{1}{2d}\sum_{x} |\nabla\varphi_x|^2 \tag{1.5}$$

where $\nabla\varphi_x := (\varphi_x - \varphi_{x+e1}, ..., \varphi_x - \varphi_{x+e_d})$. In this case, $(I - P) = -\Delta$, where

$$\Delta(x,y) = \begin{cases} -1 & \text{if } x = y, \\ \frac{1}{2d} & \text{if } |x-y| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.6}$$

is the matrix Laplacian. This is a discrete version of the usual Laplacian operator and it has analogous properties, thus for this particular model one has the whole (discrete) harmonic analysis at disposal for the investigation of the lattice free field. This together with the connection to the simple random walk makes this model particularly tracable.

1.2 Definition and basic properties

We remain in the general setting of random interface models of the last section, but we will define a Hamiltonian which is not of the form of (1.2). Let $V := [-1,1]^d$, and $V_N := NV \cap \mathbb{Z}^d$. As an operator, the discrete Laplacian Δ is defined on functions $f : \mathbb{Z}^d \to \mathbb{R}$ by

$$\Delta f(x) := \frac{1}{2d}\sum_{i=1}^{d} \left(f(x+e_i) + f(x-e_i) - 2f(x)\right).$$

It is represented by the matrix (1.6). We write (slightly abusing notation) $\Delta f_x := (\Delta f)_x = \Delta f(x)$. By Δ_N we denote the restriction of this operator to functions which are equal to 0 outside V_N. In other words, the representing matrix of Δ_N is $(\Delta(x,y))_{x,y\in V_N}$. We write Δ^2 for the iteration, $\Delta^2 f_x := \Delta^2 f(x) = \Delta(\Delta f(x))$, and Δ_N^2 for the restriction of Δ^2 to functions which are equal to 0 outside V_N. It is important to notice that $\Delta_N^2 \neq (\Delta_N)^2$. We can view Δ_N^2 as the matrix given by

$$\Delta_N^2(x,y) = \begin{cases} 1+\frac{1}{2d} & \text{if } x=y, x\in V_N, \\ -\frac{1}{d} & \text{if } |x-y|=1, x,y\in V_N, \\ \frac{1}{4d^2} & \text{if } |x-y|=2, x,y\in V_N, \\ \frac{1}{2d^2} & \text{if } |x-y|=\sqrt{2}, x,y\in V_N, \\ 0 & \text{otherwise .} \end{cases}$$

Lemma 1.2.1 *The Gibbs measure on $\mathbb{R}^{V_N}$ with $0-$boundary conditions outside V_N and Hamiltonian*

$$H_N(\varphi) = \frac{1}{2}\sum_{x\in\mathbb{Z}^d} \varphi_x \Delta^2 \varphi_x \tag{1.7}$$

exists. It is the centered Gaussian field on V_N with covariances given bycov$_N(\varphi_x,\varphi_y) = (\Delta_N^2)^{-1}(x,y)$.

Proof It is clear that Δ^2 is symmetric and positive definite. We are therefore in the setting of Proposition 13.13 of [11]. □

We call the centered Gaussian field $\{\varphi_x\}_{x\in V_N}$ on $\mathbb{R}^{V_N}$ with covariances $\mathrm{cov}_N(\varphi_x,\varphi_y) = (\Delta_N^2)^{-1}(x,y)$ the *membrane model* and denote its law by P_N. We have just seen that

$$P_N(d\varphi) = \frac{1}{Z_N}\exp\Big(-\frac{1}{2}\sum_{x\in\mathbb{Z}^d}\varphi_x\Delta^2\varphi_x\Big)\prod_{x\in V_N} d\varphi_x \prod_{x\in V_N^c}\delta_0(d\varphi_x).$$

Let for $A\subset\mathbb{Z}^d$

$$\mathcal{F}_{A^c} := \sigma(\varphi_x, x\in A^c)$$

be the sigma field generated by $\varphi_x, x\in A^c$. Recall (see [11]) that the fact that this is a Gibbs measure means that for $A\subset V_N$, the conditional distribution $P_N(\,\cdot\,|\mathcal{F}_{A^c})$, satisfies the DLR-equation

$$P_N(\,\cdot\,|\mathcal{F}_{A^c})(\psi) = P_{A,\psi}(\,\cdot\,) \quad P_N(d\psi)-a.s.,$$

where

$$P_{A,\psi}(d\varphi) := \frac{1}{Z_A}\exp\Big(-\frac{1}{2}\sum_{x\in\mathbb{Z}^d}\varphi_x\Delta^2\varphi_x\Big)\prod_{x\in A} d\varphi_x \prod_{x\in V_N\setminus A}\delta_{\psi_x}(d\varphi_x).$$

This implies that $P_N(\,\cdot\,|\mathcal{F}_{A^c})$ is the Gaussian distribution with mean

$$m_x = -\sum_{y\in A}(\Delta_A^2)^{-1}(x,y)\sum_{z\in A^c}\Delta^2(y,z)\psi_z$$

and covariance matrix $(\Delta_A^2)^{-1}$ ([11], Chapter 13). Here, Δ_A^2 is the restriction of Δ^2 to functions which are 0 outside A.
Let $G_N(x,y) := cov_N(\varphi_x,\varphi_y)$. Due to Lemma 1.2.1, we can interpret G_N as a Green's function given by the following discrete biharmonic boundary value problem on V_N : For $x\in V_N$,

$$\begin{aligned} \Delta^2 G_N(x,y) &= \delta(x,y) \qquad y\in V_N \\ G_N(x,y) &= 0, \qquad\quad\; y\in \partial_2 V_N, \end{aligned}$$

where $\partial_2 V_N := \{y\in V_N^c : 1\le \mathrm{dist}(y,V_N)\le 2\}$ is the double layer boundary of V_N. This is a discrete version of the (continuous) biharmonic boundary value problem with Dirichlet boundary conditions:

$$\begin{aligned} \Delta^2 u(x) &= f(x) \qquad x\in V \\ u(x) &= 0 \qquad\quad\; x\in\partial V \\ \tfrac{d}{dn}u(x) &= 0 \qquad\quad\; x\in\partial V. \end{aligned}$$

Here, $\frac{d}{dn}$ denotes the derivative in direction of the outer normal vector (assuming that this is well defined). We will not directly use this correspondence between discrete and continuous, apart from ganinig inspirations from standard PDE methods.
Let us stress the importance of the boundary conditions in the definition of G_N. For example, consider $\overline{G}_N(x,y) := (\Delta_N)^{-2}(x,y), x,y\in V_N$. It is easy to see that $\overline{G}_N$ is symmetric and positive definite, thus we can define the centered Gaussian field on $\mathbb{R}^{V_N}$ with covariances given by $\overline{G}_N$, but unlike the membrane model, this is not a Gibbs measure since it does not satisfy the DLR-equation. In fact, the conditional law given $\mathcal{F}_{A^c}$ in this case is the same as above, given by P_A. This means we need to understand the above model in any case. However, $\overline{G}_N$ will play an important role in our approach to the membrane model in the critical dimension, and we will investigate it in some detail in Chapter 1.4.

What is the physical interest of this model? This is better understood if we express it in a different form, using summation by parts: Assume $\varphi_x = 0$ if $x\notin V_N$, then

$$H_N(\varphi) = \frac{1}{2} \sum_{x \in \mathbb{Z}^d} (\Delta \varphi_x)^2.$$

In this form, we can compare the lattice free field and the membrane model. Obviously, the energy of the lattice free field is small if $\varphi_x \approx \varphi_y$ for nearest neighbour points x and y, which means that flat configurations are favourable. The membrane model on the other hand prefers configurations where many of the φ_x are close to the average of the height of their respective nearest neighbours. In other words, the membrane model favoures constant curvature. In the physics literature, the energy of a semiflexible membrane (or semiflexible polymer if $d = 1$) is considered to be a combination of the two:

$$H(\varphi) = \sum_x (\alpha_1 (\nabla \varphi_x)^2 + \alpha_2 (\Delta \varphi_x)^2),$$

where the parameters α_1 and α_2 are the lateral tension and the bending rigidity, respectively ([23, 14]).
The above membrane model is a special case of the following model studied in [20]: Consider the (formal) Hamiltonian

$$H'(\varphi) := \sum_{i=1}^{K} q_i \sum_x ((-\Delta)^{\frac{i}{2}} \varphi_x)^2, \tag{1.8}$$

where $K \in \mathbb{N}, q_i \in \mathbb{R}, i = 1, ..., K$, and if i is odd, we set

$$\sum_x ((-\Delta)^{\frac{i}{2}} \varphi_x)^2 := \sum_x \sum_{j=1}^{d} ((-\Delta)^{\frac{i-1}{2}} \nabla_j \varphi_x)^2,$$

where $\nabla_j(\varphi_x) := \varphi_{x+e_j} - \varphi_x$ is the first difference in the jth direction on the lattice. We will see below that the behaviour of this model depends on the minimal degree $k := \min\{i \in \mathbb{N} : q_i \neq 0\}$ of the polynomial q.
Let

$$J := \sum_{i=k}^{K} q_i (-\Delta)^i (x, y),$$

where I is the identity matrix on $\mathbb{Z}^d$. The multiplication is as before the usual matrix multiplication, or equivalently the iteration of operators.

It is well-defined since $\Delta(x,y)=0$ if $|x-y|>1$. We will always make the following assumption:

$$\sum_{i=k}^{K} q_i r^i > 0 \text{ for all } 0<r<2. \tag{1.9}$$

Note that in the case $k=K=2, q_2=1$ we have $J=\Delta^2$ and H' is equal to H, the Hamiltonian of the membrane model as defined before. For $N\in\mathbb{N}$, we define the matrix J_N analogously to Δ_N^2 as $(J(x,y))_{x,y\in V_N}$.

Lemma 1.2.2 *The Gibbs measure on $\mathbb{R}^{V_N}$ with boundary conditions ψ outside V_N and Hamiltonian (1.8) exists. It is the Gaussian field on V_N with mean*

$$m_x = -\sum_{y\in V_N} J_N^{-1}(x,y) \sum_{z\in\mathbb{Z}^d\setminus V_N} J(y,z)\psi_z,\ x\in V_N \tag{1.10}$$

and covariance matrix

$$cov_N(\varphi_x,\varphi_y) = J_N^{-1}(x,y).$$

Proof Obviously, J is translation invariant, that is, $J(x,y)=J(0,y-x)$. Let $\hat{J}(\theta) := \sum_{x\in\mathbb{Z}^d} e^{i<\theta,x>} J(0,x)$ for $\theta\in[-\pi,\pi]^d$ be the Fourier transform of J. It is immediate that $\hat{J}(\theta)=\sum_{j=k}^{K} q_j\mu(\theta)^j$, where $\mu(\theta)=\frac{1}{d}\sum_{i=1}^{d}(1-\cos\theta_i)\in[0,2]$. Thus by the Fourier inversion formula and (1.9), J is positive definite. Hence, as before, Proposition 13.13 of [11] gives the result. □

In this model, due to the Gaussianness, the parameter β of (1.1) is of no importance. We set it equal to 1. Let $G_N(x,y) := J_N^{-1}(x,y), x,y\in V_N$. Then, as before, G_N is uniquely defined as the Green's function of a discrete boundary value problem, namely

$$\begin{aligned} JG_N(x,y) &= \delta(x,y) \quad & y\in V_N \\ G_N(x,y) &= 0, & y\in\partial_K V_N. \end{aligned}$$

Here, $\partial_K V_N := \{x\in V_N^c :\ \text{dist}\ (x,V_N)\le K\}$. We will use P_N for the law of this field as well as for the special case discussed before, and refer to it as *membrane model*. It should always be made clear from the context whether we consider this general case or the special case of Δ^2. We will treat the finite volume case (critical dimension) only for $k = K = 2$. In the infinite volume case (supercritical dimensions) we consider the general model.

A basic question to ask is the existence of the infinite volume limit.

Proposition 1.2.3 *If assumption (1.9) is satisfied, the infinite volume measure*

$$P := \lim_{N\to\infty} P_N$$

exists if and only if $d \geq 2k+1$. In this case it is the centered Gaussian field on $\mathbb{Z}^d$ with covariance matrix J^{-1}.

The limit is of course taken in the weak topology on the space of probability measures.

Proof Since the P_N are all Gaussian, any limit point is necessarily Gaussian, so what we need to show is that both mean and covariances converge. From (1.10) and the fact that J has finite range, it follows immediately that for any $x \in \mathbb{Z}^d$ we have $\lim_{N\to\infty} m_x^N = 0$. As in the proof of Lemma 1.2.2, $\hat{J}(\theta) = \sum_{j=k}^K q_j \mu(\theta)^j$. Note that $\mu(\theta) = \frac{1}{2d}|\theta|^2 + o(|\theta|^2)$ as $|\theta| \to 0$. Therefore, $\hat{J}^{-1}$ is integrable if and only if $d \geq 2k+1$, and in this case by the Fourier inversion formula,

$$J^{-1}(0,x) = \frac{1}{(2\pi)^d} \int\limits_{[-\pi,\pi]^d} (\hat{J}(\theta))^{-1} e^{-i<\theta,x>} d\theta < \infty.$$

Thus if $d \geq 2k+1$, J^{-1} exists on $\mathbb{Z}^d$ and it is positive definite, since J is positive definite. $\square$

In this case, we set $G := J^{-1}$. Let $\Omega = \mathbb{R}^{\mathbb{Z}^d}$. We denote the law of this field by P. From Lemma 1.2.2 it follows, that P is the infinite volume Gibbs measure on Ω satisfying the following DLR-equation (see [11], Chapter 13):

$$P(\,\cdot\,|\mathcal{F}_{A^c}) = \mathcal{N}(\{-\sum_{y\in A} J_A^{-1}(x,y) \sum_{z\in A^c} J(y,z)\varphi_z\}_{x\in A}, J_A^{-1}) \quad P-\text{a.s.}, \tag{1.11}$$

where $J_A := (J(x,y)_{x,y\in A})$, $A \subset \mathbb{Z}^d$ a finite subset.

For the supercritical case, bounds and asymptotics on G can be obtained by purely analytical methods. Let us nevertheless give a probabilistic interpretation of G for $k = K = 2$, which will become more important in the finite volume case. Let $\mathbb{P}^x$ denote the law of a simple random walk on the lattice $\mathbb{Z}^d$ conditioned to start at $x \in \mathbb{Z}^d$. The corresponding expectation we denote by $\mathbb{E}^x$. We write $\mathbb{P}^{x,y}$ for the product measure $\mathbb{P}^x \otimes \mathbb{P}^y$, and likewise use the notation $\mathbb{E}^{x,y}$.

Proposition 1.2.4 *Let $K = k = 2$, $q_k = 1$, and $d \geq 5$. Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_m)_{m\in\mathbb{N}}$ be two independent simple random walks on $\mathbb{Z}^d$. Then*

$$G(x,y) = \mathbb{E}^{x,y}\left(\sum_{n,m}^{\infty} 1_{\{X_n = Y_m\}}\right).$$

Proof This follows immediately from the random walk representation of Δ^{-1} (compare Section 2.2 or [17], Chapter 1, for more details):

$$\begin{aligned} G(x,y) &= \sum_{z\in\mathbb{Z}^d} \Delta^{-1}(x,z)\Delta^{-1}(z,y) \\ &= \mathbb{E}^x\left(\sum_{n=0}^{\infty} 1_{\{X_n = z\}}\right) \mathbb{E}^y\left(\sum_{m=0}^{\infty} 1_{\{Y_m = z\}}\right) \\ &= \mathbb{E}^{x,y}\left(\sum_{n,m=0}^{\infty} 1_{\{X_n = Y_m\}}\right). \end{aligned}$$

□

Remark 1.2.5 *This implies $G(0,x) = \sum_{n=0}^{\infty}(n+1)p^n(0,x)$ if $k = K = 2$. Using the local central limit theorem with error bounds would be a way to obtain the asymptotic behaviour for $G(0,x)$ in the same way as for $\Delta^{-1}(0,x)$ in [17], Theorem 1.5.4. However, in the general case this is more easily obtained using Fourier transforms.*

The above proposition shows that in the infinite volume limit, the covariances are given by the local time of intersection of simple random walks. Using the methods of [18] on the convergence to local intersection times of Brownian motion, one can prove the following:

Lemma 1.2.6 *([20], Lemma 5.1) Let $d \geq 2k+1$. Then*

$$\lim_{|x|\to\infty} \frac{G(0,x)}{|x|^{2k-d}} = \frac{1}{q_k}\eta_k \,,$$

were

$$\eta_k = \frac{1}{(2\pi)^d} \int_0^\infty \int_{\mathbb{R}^d} \exp\left(i\zeta \cdot \theta - \frac{1}{(2d)^k} |\theta|^{2k} t \right) d\theta dt,$$

for any arbitrary $\zeta \in \mathbb{S}^{d-1}$.

1.3 Main results

Our main results are concerned with the effect of a "hard wall" or forbidden region on the interface. A basic object to study for random interfaces is the probability of the event that the interface is positive in a certain region (the "wall") of the lattice. Let V and V_N be defined as above. The *entropic repulsion event* is definied as

$$\Omega_N^+ := \{\varphi \in \Omega : \varphi_x \geq 0 \; \forall x \in V_N\}.$$

We think of V_N as a hard wall that forces the field to stay positive, that is, above the wall. Let us first consider the supercritical case $d \geq 2k+1$. We consider the model (1.8) and make the following additional assumption ensuring that P has the FKG-property: Let $\varepsilon > 0$. Define

$$J_\varepsilon(x,y) := \sum_{i=k}^{K} q_i (\varepsilon I - \Delta)^i (x,y).$$

Under the assumptions on H', if $d \geq 2k+1$, it is shown in [20] that J_ε^{-1} exists. Assume that there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive numbers such that

$$\lim_{n\to\infty} \varepsilon_n = 0 \text{ and } J_{\varepsilon_k}^{-1}(x,y) \geq 0 \;\; \forall k \in \mathbb{N}, \; \forall \, x,y \in \mathbb{Z}^d. \tag{1.12}$$

Note that the case $k = K = 2, q_k = 1$ satisfies this condition. Theorem 2.1 of [20] states that there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned} -C_1 &\leq \liminf_{N\to\infty} \frac{1}{N^{d-2k}\log N} \log P(\Omega_N^+) \\ &\leq \limsup_{N\to\infty} \frac{1}{N^{d-2k}\log N} \log P(\Omega_N^+) \leq -C_2 \end{aligned} \tag{1.13}$$

holds. Moreover, the constant in the lower bound has been identified as $C_1 = 2kq_k G(0,0)\mathcal{C}_k(V)$, where

$$\mathcal{C}_k(V) = \inf\left\{ \frac{1}{(2d)^k} \int_{\mathbb{R}^d} |(-\nabla)^k h|^2 dx; h \in H^k(V),\, h \geq 1 \text{ on } V \right\} \quad (1.14)$$

is the $k-$th order capacity of the unit cube V. This lower bound was proved using a relative entropy argument and the FKG-property of P. For the free field, this result was proved before in [4], where in addition was shown that the constants C_1 and C_2 of the upper and the lower bound coincide. Our first result shows that this is also true for the membrane model:

Theorem 1.3.1 *Set* $G := G(0,0)$. *If* $d \geq 2k+1$, *and assumptions (1.9) and (1.12) hold,*

$$\lim_{N\to\infty} \frac{1}{N^{d-2k}\log N} \log P(\Omega_N^+) = -2kq_k G\mathcal{C}_k(V), \quad (1.15)$$

where $C_k(V)$ *is given by* (1.14).

In Section 3.1 we will prove the upper bound in Theorem 1.3.1. Together with (1.13) and (1.14) this proves Theorem 1.3.1. Thus the decay of $P(\Omega_N^+)$ for $k \geq 2$ is completely analogous to the case $k = 1$. Knowing the decay of $P(\Omega_N^+)$, we can then address the question of the behaviour of the field conditional on the event Ω_N^+. We prove

Theorem 1.3.2 *Let* $d \geq 2k+1$ *and assume (1.9) and (1.12). Let* $\varepsilon > 0$ *and* $\eta > 0$. *Then*

$$\lim_{N\to\infty} \sup_{\substack{z\in V_N,\\ V_{N,\varepsilon}(z)\subset V_N}} P\left(\left| \frac{\overline{\varphi}_{N,\varepsilon}(z)}{\sqrt{\log N}} - \sqrt{4kG} \right| \geq \eta \,\middle|\, \Omega_N^+ \right) = 0\,, \quad (1.16)$$

where $V_{N,\varepsilon}(z) = \{x \in V_N : \max_{1\leq i\leq d} |x_i - z_i| \leq \varepsilon N\}$, *and* $\overline{\varphi}_{N,\varepsilon}(z) = \frac{1}{|V_{N,\varepsilon}(z)|} \sum_{x\in V_{N,\varepsilon}(z)} \varphi_x$.

This shows that, conditional on Ω_N^+, the local samlple mean of the field is pushed to a height of order $\sqrt{\log N}$. In the physics literature, this phenomenon is referred to as entropic repulsion ([5], [19]), since it is due to the fluctuations of the field that it moves away from the wall.

In the critical case, $d = 4$, we consider the model (1.7), that is, $k = K = 2, q_k = 1$. Let $d = 4$. We define $\gamma := \frac{8}{\pi^2}$. Our first result is the behaviour of the maximum of the field. For $0 < \delta < 1$ define

$$V_N^\delta := \{x \in V_N : \operatorname{dist}(x, V_N^c) \geq \delta N\}.$$

Theorem 1.3.3 *Let $d = 4$, and let $k = K = 2, q_k = 1$.*

(a)

$$\lim_{N\to\infty} P_N\left(\sup_{x\in V_N} \varphi_x \geq 2\sqrt{2\gamma}\log N\right) = 0$$

(b) Let $0 < \delta < 1/2$, and $0 < \eta < 1$. There exists a constant $c = c(\eta, \delta) > 0$, such that

$$P_N\left(\sup_{x\in V_N^\delta} \varphi_x \leq (2\sqrt{2\gamma} - \eta)\log N\right) \leq \exp\left(-c(\log N)^2\right).$$

These bounds on the maximum allow us to give the precise asymptotics of the probability that the field is positive on a certain region inside V_N. Let $D \subset V$ be connected with smooth boundary, which has positive distance to ∂V. Let $D_N := ND \cap \mathbb{Z}^4$ and define

$$\Omega_{D_N}^+ := \left\{\{\varphi_x\}_{x\in V_N} : \varphi_x \geq 0 \;\forall x \in D_N\right\}.$$

Here $D_N \subset V_N$ plays the role of the hard wall.

Theorem 1.3.4 *Let $d = 4$, and let $k = K = 2, q_k = 1$.*

$$\lim_{N\to\infty} \frac{1}{(\log N)^2} \log P_N(\Omega_{D_N}^+) = -8\gamma\mathcal{C}_V^2(D),$$

where $\mathcal{C}_V^2(D) = \inf\{\frac{1}{2}\int_V |\Delta h|^2 dx :\; h \in H_0(V), h \geq 1$ a.e. on $D\}$.

Concerning entropic repulsion, in the critical dimension, we can prove the following: Let $V_{\varepsilon,N}$ and $\overline{\varphi}_{\varepsilon,N}$ be defined as in Theorem 1.3.2.

Proposition 1.3.5 *Let $d = 4$, and let $k = K = 2, q_k = 1$. For any $\eta > 0$,*

$$\lim_{N\to\infty} \sup_{\substack{x\in D_N,\\ V_{\varepsilon N}(x)\subset D_N}} P_N\left(\overline{\varphi}_{\varepsilon N}(x) \le (2\sqrt{2\gamma}-\eta)\log N \,\middle|\, \Omega_N^+\right) = 0.$$

This means that the local sample mean is pushed at least to a height $2\sqrt{2\gamma}\log N$ by the wall, that is, we have again entropic repulsion. It is expected that the upper bound on the heights conditional on Ω_N^+ is of the same order.

1.4 Mathematical approach and methods

Although the gradient model and the membrane model might look similar at first sight, the mathematical methods for the investigation of the gradient model can not be used one to one for the membrane model. There are crucial differences between the Laplacian operator and its square (or any higher power). In the case of the gradient model, one has all the tools of harmonic analysis at disposition, such as the maximum principle, Harnack inequalities, and more, which provide control of many quantities involved, in particular the Green's function. Furthermore, the well-known connections between simple random walk and harmonic analysis give rise to the so-called random walk representation of the gradient model, (compare section 2.2), which is exploited in most approaches to the lattice free field (see for example [2, 4, 3, 10, 12, 22]). In our case, we lack both harmonic analysis and random walk representation.
At this point, not many mathematically rigorous results are available for the membrane model. In one dimension, pinning and wetting models are studied using a Markov renewal theory approach in [6, 7]. Entropic repulsion in the supercritcal case was approached in [20]. The results presented in this thesis are the contents of the two papers [16] (the supercritical dimensions) and [15] (critical dimension).
We have seen that in sufficiently high dimensions, the infinite volume measure exists, and we know the decay of the correlations. We even have a random walk representation for the covariances. The random

walk representation does not extend to a representation for the conditional expectations as known in the case of the lattice free fied. It is nevertheless still possible to use the methods of [4], overcoming the lack of a good expression for the expectations by a more detailed analysis of the conditional expectations using Gaussian estimates.
In the finite volume case, the representation of the covariances breaks down due to the boundary conditions we need to impose in order to get a Gibbs measure. This boundary conditions make both the use of Fourier analysis as well as a direct random walk representation impossible. However, it turns out that it is possible to compare the covariance matrix of the finite volume problem, G_N, to the Green's function $\overline{G}_N$, which is the convolution of the harmonic Green's function Γ_N with itself. $\overline{G}_N$ is therefore much easier to control and admits a random walk representation. We use a discrete version of the well-known bootstrap technique from PDE to estimate the difference between G_N and $\overline{G}_N$. In the critical and supercritical dimensions, these estimates are good enough to control the variances of the field from above and below in a manner which is sufficient to adapt the multiscale methods of [3] to prove the result on the behaviour of the maximum and on the asymptotics of $P_N(\Omega_N^+)$.
One more fact is, that for the membrane model, the FKG inequalities (see [9]) only hold for the unconditioned infinite volume measure (Corollary 1.8 of [13]). Therefore they can not be used to prove the results on entropic repulsion. This is the reason why we can only prove them for the local sample mean of the height variables, and don't obtain the upper bound on the heights in the critical case.
This thesis is organised as follows. In Chapter 2, we develop the analytical tools and prove the estimates on the variances. Some of the more technical proofs are deferred to the appendix. In Chapter 3, we prove the results on entropic repulsion for the supercritical dimensions, and in Chapter 4 those for the critical dimension.

Chapter 2
Finite volume variances

2.1 Results

In the infinite volume limit, all the information we need about the correlations of the field follow quite easily form Lemma 1.2.6. In the finite volume case, due to the boundary conditions, the problem is much more subtle, and harder to understand probabilistically, also because the representation of Proposition 1.2.4 breaks down. Before we develop the analytical methods we need, let us state the results we obtain on the finite volume variances. Throughout this section, we consider the model (1.7), that is, we assume $k = K = 2$ and $q_k = 1$. Recall the definition

$$V_N^\delta := \{x \in V_N : \mathrm{dist}(x, V_N^c) \geq \delta N\}$$

for $0 < \delta < 1$. We will prove in this section:

Proposition 2.1.1 *Let $d \geq 5$ and $0 < \delta < 1$. There exists a constant $\gamma_d > 0$ such that*

(a) $\sup_{x \in V_N} var_N(\varphi_x) \leq \gamma_d + O(N^{4-d})$.

(b) $\sup_{x \in V_N^\delta} |var_N(\varphi_x) - \gamma_d| = O(N^{4-d})$.

In the critical dimension, we have a logarithmic correction.

Proposition 2.1.2 *Let $d = 4$ and $0 < \delta < 1$. Let $\gamma = \frac{8}{\pi^2}$.*

(a) There exists $C > 0$ such that $\sup_{x \in V_N} var_N(\varphi_x) \leq \gamma \log N + C$.

(b) There exists $C(\delta) > 0$ such that $\sup_{x \in V_N^\delta} |var_N(\varphi_x) - \gamma \log N| \leq C(\delta)$.

In the subcritical case, our methods work less well and provide only upper bounds on the variances. This is one (but not the only) reason why we don't obtain satisfactory results for the membrane model in dimension 2 and 3. Note that of course in dimension 1, one can get explicit expressions by direct computation.
To prove Propositions 2.1.1 and 2.1.2 we need to control $G_N(x,y)$. To this purpose, we compare it to a biharmonic Green's function with different boundary conditions. Let

$$E_1 := \{v : V_N \cup \partial_2 V_N \to \mathbb{R} : v(x) = 0 \; \forall x \in \partial_2 V_N\}.$$

Recall from the introduction that the covariance matrix of the membrane model is given by the unique function $G_N(x,\cdot)$ in E_1 which satisfies $\Delta^2 G_N(x,y) = \delta(x,y)$.
Let us introduce the usual harmonic Green's function. Let A be an arbitrary subset of $\mathbb{Z}^d$, fix $x \in A$, and let $\Gamma_A(x,\cdot)$ be the unique lattice function which satisfies

$$\begin{array}{rcll} \Delta\Gamma_A(x,y) & = & -\delta(x,y) & y \in A, \\ \Gamma_A(x,y) & = & 0 & y \in \partial A. \end{array}$$

(Existence and uniqueness follow from standard discrete harmonic analysis, see for example Chapter I of [17]). Let $\Gamma_N(x,y) := \Gamma_{V_N}(x,y)$. Define now for $x, y \in V_N$,

$$\overline{G}_N(x,y) := \sum_{z \in V_N} \Gamma_N(x,z)\Gamma_N(z,y),$$

and extend $\overline{G}_N(x,\cdot)$ to a function on $V_N \cup \partial_2 V_N$ by requiring

$$\begin{array}{rcll} \overline{G}_N(x,y) & = & 0 & y \in V_{N+1} \setminus V_N, \text{ and} \\ \Delta\overline{G}_N(x,y) & = & 0 & y \in \partial V_N. \end{array}$$

For $y \in \partial V_{N+1}$, this means defining $\overline{G}_N(x,y) := -\overline{G}_N(x,\tilde{y})$, where $\tilde{y}$ is the unique point in V_N with $dist(y,\tilde{y}) = 1$. It is straightforward to check that then $\Delta^2\overline{G}_N(x,y) = \delta(x,y)$ for all $x, y \in V_N$. In fact, $\overline{G}_N(x,\cdot)$ is the (again unique) function which satisfies

$$\begin{array}{rcll} \Delta^2\overline{G}_N(x,y) & = & \delta(x,y) & y \in V_N, \\ \overline{G}_N(x,y) & = & 0 & y \in V_{N+1} \setminus V_N, \\ \Delta\overline{G}_N(x,y) & = & 0 & y \in \partial V_N. \end{array}$$

The main idea of this section is to compare $G_N(x,y)$ and $\overline{G}_N(x,y)$. An upper bound (2.1.1(a) and 2.1.2(a)) is obtained by showing that $G_N(x,x) \leq \overline{G}_N(x,x)$ for all $x \in V_N$ and bounding $\overline{G}_N(x,x)$, using the well-known facts on Γ_N. To prove part (b) of the Propositions, we will later on show that if $x \in V_N^\delta$,

$$\sup_{y \in V_N^\delta} |G_N(x,y) - \overline{G}_N(x,y)| \leq c$$

for some $c = c(\delta) < \infty$. This will be done by studying the boundary value problem satisfied by $G_N(x,y) - \overline{G}_N(x,y)$ and showing that the solution of this boundary value problem is sufficiently regular (in a sense to be specified). Since $\overline{G}_N$ is given in terms of Γ_N, well-known results from harmonic analysis and random walks give us a very good control on the behaviour of $\overline{G}_N(x,y)$. Combining all this will then prove Propositions 2.1.1 and 2.1.2.

2.2 Green's function of the simple random walk

Before embarking on the comparison of G_N and $\overline{G}_N$, we derive the necessary estimates on $\overline{G}_N$. We collect some results on Γ_N, which we will use to describe $\overline{G}_N$. For all the proofs we omit here, we refer to [17]. Let $(X_n)_{n\in\mathbb{N}}$ be a simple random walk on $\mathbb{Z}^d$. As before, we use the notation $\mathbb{P}^x$ and $\mathbb{E}^x$ respectively for the law and expectation conditional on $X_0 = x$. Let $A \subset \mathbb{Z}^d$ be finite. We let

$$\tau_A = \inf\{n \geq 0 : X_n \notin A\}$$

denote the first exit time of A. It is a well-known fact that $\Gamma_A(x,y), x, y \in \mathbb{Z}^d$, is given by

$$\Gamma_A(x,y) = \mathbb{E}^x\left(\sum_{n=0}^{\tau_A - 1} 1_{\{X_n = y\}}\right) = \sum_{n=0}^{\infty} \mathbb{P}^x(X_n = y, n < \tau_A).$$

Obviously $\Gamma_A(x,y) = 0$ if $x \notin A$ or $y \notin A$. By the reversibility of simple random walk, $\Gamma_A(x,y) = \Gamma_A(y,x)$. Also, if $B \subset A$, we have $\Gamma_B(x,y) \leq$

$\Gamma_A(x,y)$. If $d \geq 3$, simple random walk is transient, which implies that

$$\Gamma(x,y) := \lim_{N\to\infty} \Gamma_N(x,y) = \mathbb{E}^x\left(\sum_{n=0}^{\infty} 1_{\{X_n=y\}}\right) < \infty \quad \forall x,y \in \mathbb{Z}^d.$$

It is clear that $\Gamma(x,y) = \Gamma(0,x-y)$, and of course Γ is symmetric as well. Obviously

$$\Delta\Gamma(0,x) = -\delta(x).$$

Thus $-\Gamma$ is the Green's function (or fundamental solution) of the discrete Laplacian, as Γ_A is the Green's function of the discrete harmonic Dirichlet problem on A.

Let $p_n(x,y) := \mathbb{P}^x(X_n = y)$ denote the n−step transition probability of simple random walk on $\mathbb{Z}^d$. Define

$$\overline{p}_n(x) := 2\left(\frac{d}{2\pi n}\right)^{d/2} \exp(-d|x|^2/(2n))$$

and

$$E(n,x) := \begin{cases} p_n(0,x) - \overline{p}_n(x) & \text{if } x \text{ and } n \text{ have the same parity,} \\ 0 & \text{otherwise.} \end{cases}$$

The local central limit theorem with error bounds gives estimates on $E(n,x)$:

Lemma 2.2.1 *([17], Theorem 1.2.1) With the above definitions,*

$$|E(n,x)| \leq O(n^{-(d+2)/2})$$

and

$$|E(n,x)| \leq |x|^{-2} O(n^{-d/2}).$$

This gives the following estimate on $\Gamma(0,x)$:

Lemma 2.2.2 *([17], Theorem 1.5.4) If $d \geq 3$, as $|x| \to \infty$,*

$$\Gamma(0,x) \sim a_d |x|^{2-d},$$

where $a_d = \frac{2}{(d-2)\omega_d}$ and $\omega_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$ is the volume of the unit ball in $\mathbb{R}^d$.

Lemma 2.2.3 *([17], Proposition 1.5.8) If $d \geq 3$, then*

$$\Gamma_A(x,y) = \Gamma(x,y) - \sum_{z \in \partial A} \mathbb{P}^x(X_{\tau_A} = z)\Gamma(z,y).$$

We will often use the following:

Lemma 2.2.4 *([17], Proposition 1.5.9) Let B_N be the ball with radius N about 0. If $d \geq 3$,*

$$\Gamma_{B_N}(0,x) = a_d\left(\frac{1}{|x|^{d-2}} - \frac{1}{N^{d-2}}\right) + O(|x|^{1-d}).$$

Denote the first difference in the ith direction of a function $v : \mathbb{Z}^d \to \mathbb{R}$ by $\nabla_i v(x) := v(x+e_i) - v(x)$, and more general for a multiindex $\alpha = (\alpha_1, ..., \alpha_k) \in \mathbb{N}^k$ write $\nabla^\alpha v(x) := \nabla_1^{\alpha_1} ... \nabla_d^{\alpha_k} v(x)$. By $D^\alpha f$ we denote the usual partial derivative of a function $f : \mathbb{R}^d \to \mathbb{R}$. We will need the following generalisation of Theorem 1.5.5 of [17].

Lemma 2.2.5 *If $d \geq 3$, for any multiindex α and any $x \in \mathbb{Z}^d$, with $|x| > |\alpha|$,*

$$\nabla^\alpha \Gamma(0,x) - a_d D^\alpha(|x|^{2-d}) = O(|x|^{-d-|\alpha|+1}).$$

Proof For $y \in \mathbb{Z}^d$ and any function f on $\mathbb{Z}^d$ define

$$\nabla_y f(x) := f(x+y) - f(x).$$

Assume that y is even. As in the proof of Theorem 1.2.1 of [17] we can write

$$E(n,x) = J(n,x) + \frac{2}{(2\pi\sqrt{n})^{d/2}} I(n,x),$$

for some $J(n,x)$ with $|J(n,x)| \leq \varrho^n$ for some $0 < \varrho < 1$, and

$$I(n,x) = \int\limits_{|\theta| \leq n^{1/4}} \left(\varphi^n(|\theta|/\sqrt{n}) - \exp(-|\theta|^2/(2d)\right) \exp(-(ix\cdot\theta)/\sqrt{n}) d\theta$$

where $\varphi^n(|\theta|/\sqrt{n}) = \exp(-|\theta|^2/(2d))(1 + |\theta|^8 O(n^{-1}))$. It is easy to see that for even $y_1, ..., y_k$

$$\nabla_{y_1} ... \nabla_{y_k} \exp\left(\frac{-ix\cdot\theta}{\sqrt{n}}\right) = O(n^{-k/2}) \exp\left(\frac{-ix\cdot\theta}{\sqrt{n}}\right),$$

which implies $|\nabla_{y_1}...\nabla_{y_k}I(n,x)| = c(y)O(n^{k/2+1})$ and thus

$$|\nabla_{y_1}...\nabla_{y_k}E(n,x)| \le c(y)O(n^{(d+2+k)/2}),$$

and as on page 17 of [17],

$$|\nabla_{y_1}...\nabla_{y_k}E(n,x)| \le c(y)|x|^{-2}O(n^{(d+k)/2}).$$

Now it follows in a completely analagous way as in the proof of Lemma 1.5.2 of [17], that for even $y_1,...,y_k$, and any $\alpha < d$,

$$\lim_{|x|\to\infty} |x|^{\alpha+k}\sum_{j=0}^{\infty}|\nabla_{y_1}...\nabla_{y_k}E(j,x)| = 0. \tag{2.17}$$

If $y_l = |y_l|u_l$, where $|u_l| = 1$ $(1 \le l \le k)$, and $|x| \ge 2|y|$ we have by the Taylor formula with remainder,

$$\begin{aligned}|\nabla_{y_1}...\nabla_{y_k}\exp(-|x|^2/(4n)) - |y|D_{u_1}...D_{u_k}\exp(-|x|^4/(n))| \\ \le c(y)\exp(-|x|^2/(16n))(|x|^{k+1}O(n^{-k-1}) + |x|^{k-1}O(n^{-k})).\end{aligned}$$

Furthermore

$$\begin{aligned}\sum_{n=0}^{\infty} 2\left(\frac{d}{4\pi n}\right)^{d/2} |y|D_{y_1}...D_{y_k}\exp\left(-(d|x|^2)/(4n)\right) \\ = |y|D_{y_1}...D_{y_k}(a_d|x|^{2-d}) + O(|x|^{-d-2}).\end{aligned}$$

Since

$$\nabla_{e_i}\Gamma(0,x) = \frac{1}{2d}\sum_{|e|=1}\nabla_{e_i+e}\Gamma(0,x),$$

and similarly for ∇^α for general α, together with (2.17) this proves

$$\nabla^\alpha\Gamma(0,x) = a_d D^\alpha(|x|^{2-d}) + O(|x|^{-d-|\alpha|+1})$$

even x. If x is odd, we have

$$\begin{aligned}\nabla^\alpha\Gamma(0,x) = &\ \tfrac{1}{(2\pi)^d}\textstyle\sum_{|f_1|=1}\cdots\sum_{|f_d|=1}\nabla^\alpha\Gamma(x+f) \\ = &\ D^\alpha(a_d|x|^{2-d}) + O(|x|^{-d-|\alpha|+1}).\end{aligned}$$

. $\square$

2.3 Estimates on $\overline{G}_N$

In this section, we derive estimates on $\overline{G}_N(x,y)$, using the fact that $\overline{G}_N$ is just the convolution of Γ_N with itself. This immediately leads to a representation of $\overline{G}_N$ in terms of simple random walk which is analogous to the one for the infinite volume covariance matrix in Proposition 1.2.4. Let $(X_k),(Y_m)$ be two independent simple random walks on the lattice $\mathbb{Z}^d$, and let τ_N as before denote the first exit time of V_N.

Lemma 2.3.1 *If $x,y \in V_N$ the following hold:*

$$\overline{G}_N(x,y) = \mathbb{E}^{x,y}\left[\sum_{k=0}^{\tau_N-1}\sum_{m=0}^{\tau_N-1} 1_{\{X_k=Y_m\}}\right] = \sum_{k=0}^{\infty}(k+1)\mathbb{P}^x(X_k=y, k<\tau_N).$$

Proof We have for $x,y\in V_N$

$$\overline{G}_N(x,y) = \sum_{z\in V_N}\Gamma_N(x,z)\Gamma_N(z,y) = \mathbb{E}^{x,y}\left[\sum_{k=0}^{\tau_N-1}\sum_{m=0}^{\tau_N-1} 1_{\{X_k=Y_m\}}\right],$$

and

$$\begin{aligned}\overline{G}_N(x,y) &= \sum_{z\in V_N}\Gamma_N(x,z)\Gamma_N(z,y)\\ &= \sum_{k,m=0}^{\infty}\sum_{z\in V_N}\mathbb{P}^x(X_k=z,k<\tau_N)\mathbb{P}^z(Y_m=y,m<\tau_N)\\ &= \sum_{k,m=0}^{\infty}\mathbb{P}^x(X_{k+m}=y,k+m<\tau_N) = \sum_{k=0}^{\infty}(k+1)\mathbb{P}^x(X_k=y,k<\tau_N).\end{aligned}$$

This proves the lemma. □

Estimates on $\overline{G}_N(x,x)$ are easily obtained from the estimates on $\Gamma_N(x,y)$:

Lemma 2.3.2 *Let $d=4$ and let $\delta\in(0,1/2)$. There exist constants $C_1>0$, $C_2=C_2(\delta)>0$, such that*

(a)

$$\sup_{x\in V_N}\overline{G}_N(x,x) \le \frac{8}{\pi^2}\log N + C_1,$$

(b)

$$\inf_{x\in V_N^\delta} \overline{G}_N(x,x) \geq \frac{8}{\pi^2}\log N + C_2.$$

Proof Let $B_r(x)$ denote the ball of radius r and centre x. Since we have $\Gamma_N(x,x) \leq \Gamma(x,x)$, we obtain, approximating the sum by the integral und using the well-known result about integrals of rotationally symmetric functions,

$$\begin{aligned}\overline{G}_N(x,x) \leq & \sum_{z\in B_{2N(x)}} \Gamma(x,z)\Gamma(z,x) \leq a_4^2 \sum_{\substack{z\in B_{2N}(x)\\ z\neq x}} \frac{1}{|x-z|^4} + O(1)\\ \leq & 4a_4^2\omega_4 \int_1^{2N} \frac{1}{r}dr + O(1) = \frac{8}{\pi^2}\log(2N) + C.\end{aligned}$$

Here, ω_4 is the volume of the unit ball in $\mathbb{R}^4$ and a_4 is the constant in (2.2.2). The lower bound follows by taking $B_{\delta N}(x)$ in the place of $B_{2N}(x)$:

$$\begin{aligned}\overline{G}_N(x,x) \geq & \sum_{z\in B_{\delta N}} \Gamma_{B_{\delta N}}(x,z)\Gamma_{B_{\delta N}}(z,x) \geq 4a_4^2\omega_4 \int_1^{\delta N} \frac{1}{r}dr + O(1)\\ = & \frac{8}{\pi^2}\log(\delta N) + C(\delta).\end{aligned}$$

□

Similarly for $d \geq 5$:

Lemma 2.3.3 *Let $d \geq 5$ and let $\delta \in (0, 1/2)$. There exist constants γ_d and $C(\delta) = C(\delta, d) > 0$, such that*

(a)

$$\sup_{x\in V_N} \overline{G}_N(x,x) \leq \gamma_d + O(N^{4-d})$$

(b)

$$\inf_{x\in V_N^\delta} \overline{G}_N(x,x) \geq \gamma_d + C(\delta)N^{4-d}.$$

Proof As before,

$$\overline{G}_N(x,x) \le \sum_{z\in V_N} \Gamma(x,z)\Gamma(z,x) \le a_d^2 \sum_{\substack{z\in B_{2N}(x)\\ z\neq x}} \frac{1}{|x-z|^4} + \Gamma(0,0)^2$$

$$\le d a_d^2 \omega_d \int_1^{2N} \frac{1}{r^{d-3}} dr + O(N^{3-d}) + \Gamma(0,0)^2 = \gamma_d + O(N^{4-d}),$$

where $\gamma_d = d a_d^2 \omega_d (d-4) + \Gamma(0,0)^2 = \frac{d^2(d+2)\Gamma(d/2+1)}{\pi^{(d/2)}} + \Gamma(0,0)^2$. (Note that $\Gamma(\cdot)$ with one argument denotes the Gamma function, as opposed to the Green's function $\Gamma(\cdot,\cdot)$ with two arguments). For the lower bound, note that $\Gamma_N(0,0) = \Gamma(0,0) + O(N^{2-d})$. Then

$$\overline{G}_N(x,x) \ge \sum_{z\in B_{\delta N}} \Gamma_{B_{\delta N}}(x,z)\Gamma_{B_{\delta N}}(z,x) + \Gamma_{B_\delta N}(0,0)^2$$

$$\ge d a_d^2 \omega_d \int_1^{\delta N} \frac{1}{r^{d-3}} dr + \Gamma(0,0)^2 + O(N^{3-d})$$

$$= c_d + \Gamma(0,0)^2 + C(\delta) N^{4-d}.$$

□

The fact that far from the singularity, the Green's function $\overline{G}_N$ is smooth will play an important role to prove the lower bound on the variances.

Lemma 2.3.4 *Let $d \ge 4$. Let $0 < \delta < 1/2$, and $0 < \delta' < \delta/2$, and let $x \in V_N^\delta$. There exists a constant $c = c(\delta) > 0$ such that for all y with $\frac{\delta'}{2}N \le dist(y, V_N^c) \le \delta' N$, for all $0 \le |\alpha| \le 4$*

$$|\nabla^\alpha \overline{G}_N(x,y)| \le \frac{c}{N^{d+|\alpha|-4}}.$$

Proof We may assume $x = 0$. By Lemma 2.2.5,

$$\nabla^\alpha \Gamma(0,y) = a_d D^\alpha(|y|^{2-d}) + O(|y|^{-d-|\alpha|+1})$$

for some constant a_d. Since $\Gamma_N(x,y) = \Gamma(x,y) - \sum_{z\in\partial V_N} \mathbb{P}^0(X_{\tau_N} = z)\Gamma(z,y)$, it follows immediately that for any y with $dist(y,\partial V_N) \ge \delta' N$ and $|x-y| \ge (\delta/2)N$ we have

$$|\nabla^\alpha \Gamma_N(x,y)| \le c(\delta,\delta') N^{-d-|\alpha|+2}.$$

We first assume $x = 0$. Split

$$\begin{aligned}\nabla^\alpha \overline{G}_N(0,y) &= \sum_{z\in V_N} \Gamma_N(0,z)\nabla^\alpha\Gamma_N(z,y)\\ &= \sum_{z\in V_N^\delta} \Gamma_N(0,z)\nabla^\alpha\Gamma_N(z,y) + \sum_{z\in V_N\setminus V_N^\delta} \Gamma_N(0,z)\nabla^\alpha\Gamma_N(z,y).\end{aligned}$$

If $z \in V_N^\delta$ and $\mathrm{dist}(y, V_N^c) \geq \delta' N$, we have $|z-y| \geq \delta' N$, and we can bound the first term by

$$\Big|\sum_{z\in V_N^\delta} \Gamma_N(0,z)\nabla^\alpha\Gamma_N(z,y)\Big| \leq \frac{c}{N^{d+|\alpha|-2}} \sum_{z\in V_N^\delta} \frac{1}{|z|^{d-2}} \leq \frac{c}{N^{d+|\alpha|-4}}.$$

The second term we split again:

$$\begin{aligned}&\sum_{z\in V_N\setminus V_N^\delta} \Gamma_N(0,z)\nabla^\alpha\Gamma_N(z,y)\\ &\qquad= \sum_{z\in V_N\setminus V_N^\delta} \Gamma_N(0,z)\nabla^\alpha\Gamma(z,y)\\ &\qquad\quad - \sum_{z\in V_N\setminus V_N^\delta}\sum_{w\in\partial V_N} \mathbb{P}^z(X_{\tau_N}=w)\Gamma_N(0,z)\nabla^\alpha\Gamma(w,y).\end{aligned}$$

Again we have for any $w \in \partial V_N$ that $|w-y| \geq \delta' N$ and therefore as above

$$\Big|\sum_{z\in V_N\setminus V_N^\delta}\sum_{w\in\partial V_N} \mathbb{P}^z(X_{\tau_N}=w)\Gamma_N(0,z)\nabla^\alpha\Gamma(w,y)\Big| \leq cN^{-d-|\alpha|+4}.$$

For the remaining term we use summation by parts (for $|\alpha| \leq 2$ this is not necessary, we could use similar estimates as before). Note that since $\Gamma(z,y) = \Gamma(y,z)$ we have

$$\Gamma(z, y+e_i) - \Gamma(z,y) = \Gamma(z-e_i, y) - \Gamma(z,y)$$

and thus

$$\nabla^\alpha\Gamma(z,y) = \nabla^{-\alpha}\Gamma(y,z)$$

(we always let the difference operator act on the second variable). Thus if $\alpha = \alpha' + e_i$, by summation by parts

$$\begin{aligned}\sum_{z \in V_N \setminus V_N^\delta} \Gamma_N(0,z)\nabla^\alpha \Gamma(z,y) = & \sum_{z \in V_N \setminus V_N^\delta} \nabla^{e_i}\Gamma_N(0,z)\nabla^{\alpha'}\Gamma(z,y) \\ & + \sum_{z \in \partial(V_N \setminus V_N^\delta)} r(z)\Gamma_N(0,z)\nabla^{\alpha'}\Gamma(z,y)\end{aligned}$$

where $1 \le r(z) \le d$ is the number of points in $V_N \setminus V_N^\delta$ which are neighbours of z. Note that

$$\begin{aligned}\sum_{z \in \partial(V_N \setminus V_N^\delta)} r(z)\Gamma_N(0,z)\nabla^{\alpha'}\Gamma(z,y) &\le cN^{d-1}\frac{1}{N^{d-2}}\frac{1}{N^{d+|\alpha'|-2}} \\ &\le c\frac{1}{N^{d+|\alpha|-4}}.\end{aligned}$$

Similarly we have for any α', β with $|\alpha'| + |\beta| = |\alpha| - 1$ that

$$\sum_{z \in \partial(V_N \setminus V_N^\delta)} r(z)\nabla^\beta\Gamma_N(0,z)\nabla^{\alpha'}\Gamma(z,y) \le c\frac{1}{N^{d+|\alpha|-4}}.$$

Hence we can iterate summation by parts and obtain that

$$\begin{aligned}&\Big|\sum_{z \in V_N \setminus V_N^\delta} \Gamma_N(0,z)\nabla^\alpha\Gamma(z,y)\Big| \\ &\qquad \le \Big|\sum_{z \in V_N \setminus V_N^\delta} \nabla^\alpha\Gamma_N(0,z)\Gamma(z,y)\Big| + c\frac{1}{N^{d+|\alpha|-4}} \\ &\qquad \le c\frac{1}{N^{d+|\alpha|-2}}\sum_{z \in V_N \setminus V_N^\delta}\frac{1}{|z-y|^{d-2}} + c\frac{1}{N^{d+|\alpha|-4}} \\ &\qquad \le \frac{c}{N^{d+|\alpha|-4}}.\end{aligned}$$

This completes the proof, since similar arguments hold if $x \in V_N^\delta$ is arbitrary. $\square$

2.4 Upper bound on the variances

Recall $\overline{G}_{N+1}(x,y) = \sum_{z\in V_{N+1}} \Gamma_{N+1}(x,z)\Gamma_{N+1}(z,y)$, which is defined for $x \in V_{N+1}, y \in V_{N+1} \cup \partial V_{N+1}$. Let $\tilde{H}_N(x,y) := \overline{G}_{N+1}(x,y) - G_N(x,y)$. We show that $\tilde{H}_N(x,x) \geq 0$ for all $x \in V_N$. Let $E_N := \{h : V_N \cup \partial_2 V_N \to \mathbb{R} : \Delta h(x) = 0 \; \forall x \in V_N\}$. This is a linear subspace of the finitely dimensional vector space $\mathbb{R}^{V_N \cup \partial_2 V_N}$. Let k_N denote the orthogonal projection from $\mathbb{R}^{V_N\cup\partial_2 V_N}$ onto E and let $k_N(x,y)$ be the representing matrix of k_N with respect to the standard basis of $\mathbb{R}^{V_N\cup\partial_2 V_N}$.

Lemma 2.4.1
For all $x, y \in V_N$,

$$\tilde{H}_N(x,y) = \sum_{w,v\in V_{N+1}} \Gamma_{N+1}(x,w)k_N(w,v)\Gamma_{N+1}(v,y).$$

Proof It is clear that for fixed $x \in V_N$, $\tilde{H}_N$ is uniquely defined as the solution of the discrete boundary value problem

$$\begin{array}{rcll} \Delta^2 \tilde{H}_N(x,y) &=& 0 & y \in V_N \\ \tilde{H}_N(x,y) &=& \overline{G}_N(x,y) & y \in \partial_2 V_N = (V_{N+1}\setminus V_N) \cup \partial V_{N+1}. \end{array}$$

Write $K_N(x,y) := \sum_{w,v\in V_{N+1}} \Gamma_{N+1}(x,w)k_N(w,v)\Gamma_{N+1}(v,y)$. To prove the lemma, it is sufficient to show that K_N satisfies the same boundary value problem as $\tilde{H}_N$. By definition of k_N it is clear that $\Delta^2 K_N(x,y) = \Delta(k_N\Gamma_N)(x,y) = 0$ if $y \in V_N$. Also, $K_N(x,y) = 0 = \tilde{H}_N(x,y)$ if $y \in \partial V_{N+1}$, since in this case $\Gamma_{N+1}(x,y) = 0$. If $y \in V_{N+1} \setminus V_N$ is fixed, define the function $\alpha : V_N \cup \partial_2 V_N \to \mathbb{R}$ by $\alpha(z) := \Gamma_{N+1}(z,y)$. Then, since $y \in V_{N+1}\setminus V_N$, we see that $\Delta\alpha(z) = 0$ for all $z \in V_N$, and therefore by definition of k_N as a projection, $k_N\Gamma_{N+1}(w,y) = \Gamma_{N+1}(w,y))$ for all $w \in V_{N+1}$. Thus $K_N(x,y) = \overline{G}_{N+1}(x,y) = \tilde{H}_N(x,y)$ in this case. This shows $K_N = \tilde{H}_N$. $\square$

Proof of Proposition 2.1.1 (a) and Proposition 2.1.2(a). Since k_N is an orthogonal projection, we have $k_N^2 = k_N$ and $k_N(v,w)$ is symmetric. Let $k_N^2(w,v) = \sum_t k_N(w,t)k_N(t,v)$. Then for all $x \in V_N$,

$$\begin{aligned} \tilde{H}_N(x,x) = K_N(x,x) &= \sum_{w,v\in V_{N+1}} \Gamma_{N+1}(x,w)k_N^2(w,v)\Gamma_{N+1}(v,x) \\ &= \langle k_N^2\Gamma_{N+1}(\cdot,x), \Gamma_{N+1}(\cdot,x)\rangle \geq 0 \end{aligned}$$

since k_N^2 is positive definite. But this implies

$$G_N(x,x) = \overline{G}_{N+1}(x,x) - \tilde{H}_N(x,x) \le \overline{G}_{N+1}(x,x).$$

Together with the upper bounds on $\overline{G}_{N+1}$ of Lemma 2.3.2 and Lemma 2.3.3 (which do not depend on δ) this proves the upper bound on the variances.

2.5 Lower bound on the variances

Our setting is discrete, and therefore a problem of linear algebra. However, the ideas behind our strategy are inspired by standard methods from the theory of partial differenital equations, and we use the PDE-terminology. We need to introduce discrete Sobolev norms. Let $\partial_- V_N := \{x \in V_N : \mathrm{dist}(x, V_N^c) \le 1\}$. For a function $v : V_N \cup \partial_k V_N \to \mathbb{R}$ define

$$\|v\|^2_{H^k(V_N)} := \sum_{j=0}^{k} \sum_{\substack{\alpha\in\mathbb{N}^d:\\ |\alpha|=j}} \sum_{x\in V_N} \left(N^j \nabla^\alpha v(x)\right)^2.$$

For $v, w \in E_1$ define

$$\mathcal{D}(v,w) := \sum_{x\in V_N} \Delta v(x)\Delta w(x) + \sum_{x\in\partial_- V_N} r(x)v(x)w(x)$$

where $r(x) := |\{y \in V_N^c : \mathrm{dist}(x,y) = 1\}|$. Obviously, $1 \le r(x) \le d$ for all $x \in \partial_- V_N$. It is immediate that $\mathcal{D}(\cdot,\cdot)$ is symmetric, bilinear and positive definite. We write $\|v\|_{\mathcal{D}} := \sqrt{\mathcal{D}(v,v)}$. In Appendix A.1, we prove some estimates for discrete Sobolev norms and the Dirichlet form $\mathcal{D}(\cdot,\cdot)$.

To compare G_N and $\overline{G}_N$, we use the fact that the difference of the two Green's functions, $H_N(x,y) := \overline{G}_N(x,y) - G_N(x,y)$, satisfies the following boundary value problem:

$$\begin{aligned} \Delta^2 H_N(x,y) &= 0 & y &\in V_N \\ H_N(x,y) &= \overline{G}_N(x,y) & y &\in \partial_2 V_N. \end{aligned}$$

Let f be any function $V_N \cup \partial_2 V_N \to \mathbb{R}$ which satisfies $f(y) = \overline{G}_N(x,y)$ for all $y \in \partial_2 V_N$. Then $u(y) := H_N(x,\cdot) - f(\cdot)$ satisfies

$$\begin{aligned} \Delta^2 u(y) &= g(y) & y &\in V_N \\ u(y) &= 0 & y &\in \partial_2 V_N, \end{aligned} \tag{2.18}$$

where $g(y) = -\Delta^2 f(y)$. The idea is to choose f sufficiently regular in the interiour of V_N, and show that this yields a solution u of (2.18) which has the same regularity on V_N^δ. Then we can derive estimates on $H_N(x,y)$ for $x, y \in V_N^\delta$.
Note that a function u is a solution of (2.18) if and only if for any function $v : V_N \cup \partial_2 V_N \to \mathbb{R}$ it satisfies

$$\sum_{x \in V_N} \Delta^2 u(x) v(x) = \sum_{x \in V_N} g(x) v(x).$$

(Take $v = 1_x, x \in V_N$ to prove the "only if "-direction). Summation by parts now shows that, since $u(x) = 0$ for all $x \in \partial_2 V_N$,

$$\sum_{x \in V_N} \Delta^2 u(x) v(x) = \mathcal{D}(u, v).$$

Hence $\mathcal{D}(\cdot, \cdot)$ is the Dirichlet form corresponding to our boundary value problem, and therefore an equivalent formulation of (2.18) is

$$\mathcal{D}(u, v) = \langle g, v \rangle_{L_2(V_N)} \quad \forall v \in E_1, \tag{2.19}$$

where $\langle \cdot, \cdot \rangle_{L_2(V_N)}$ denotes the L_2 scalar product on V_N. The Riesz Theorem now gives us a "weak" solution of (2.19): Clearly, for fixed $w \in E_1$ the map $v \mapsto \mathcal{D}(v, w)$ is well defined and linear from $E_1 \to \mathbb{R}$, so that by Riesz there exists $h_w \in E_1$ such that $\mathcal{D}(v, w) = \langle h_w, v \rangle_{L_2(V_N)}$, and the map $A : w \mapsto h_w$ is well defined and linear. It is injective, and therefore bijective since E_1 is finite dimensional. Thus A^{-1} exists, and $u := A^{-1}(-\Delta^2 f)$ is a solution of (2.19) and therefore also a solution of (2.18).

Lemma 2.5.1 *The unique solution u of (2.18) satisfies $\|u\|_{H^2(V_N)} \le cN^4 \|g\|_{L_2(V_N)}$.*

Proof We have just shown existence and uniqueness. For the norm estimate, note that by Corollary A.1.6, $\|u\|^2_{H^2(V_N)} \le cN^4 \mathcal{D}(u, u) = cN^4 \langle g, u \rangle_{L^2(V_N)} \le cN^4 \|g\|_{L^2(V_N)} \|u\|_{L_2(V_N)}$. This implies $\|u\|_{H^2(V_N)} \le cN^4 \|g\|_{L^2(V_N)}$. $\square$

For our purpose, we need stronger regularity of the solution than what we obtain from Lemma 2.5.1. To obtain this, we use a discrete version of the well-known bootstrap-technique in PDE, compare for example Chapter 20 of [24]. The first step is the following Lemma.

Lemma 2.5.2 *Let $1/2 < \delta < 1$, $0 < \varepsilon < 1/8$, and let N be large enough, such that $\varepsilon N > 1$. Let $\chi : \mathbb{Z}^d \to \mathbb{R}$ satisfy $|\nabla^\alpha \chi| \le cN^{-|\alpha|}$ for any multiindex α, $\chi = 1$ on V_N^δ and $\chi(x) = 0$ if $dist(x, \partial V_N) \le 2\varepsilon N$. Furthermore, let $v : V_N \to \mathbb{R}$ be any function with $v(x) = 0$ if $dist(x, \partial V_N) \le \varepsilon N$. Then there exists $\overline{v}$ with $\|\overline{v}\|_{H^2(V_N)} = \|v\|_{H^2(V_N)}$, such that*

$$N^4 \mathcal{D}(N\nabla_i(\chi u), v) = -N^4 \langle g, N\chi\nabla_i \overline{v}\rangle_{L_2(V_N)} + I_0,$$

where $I_0 \le c\|u\|_{H^2(V_N)} \|v\|_{H^2(V_N)}$.

Proof First, note the product rule for $\nabla_i : \nabla_i(vw)(x) = \nabla_i v(x) w(x) + v(x+e_i)\nabla_i w(x)$. Furthermore, if v has support in the interior of V_N, then $\sum_{x\in V_N} \nabla_i v(x) = 0$. Using this and the assumptions on v, we get

$$\begin{aligned} N^4 \mathcal{D}(N\nabla_i(\chi u), v) =& N^4 \sum_{x\in V_N} \Delta N \nabla_i(\chi u)(x) \Delta v(x) \\ =& N^4 \sum_{x\in V_N} N\nabla_i \Delta(\chi u)(x)\Delta v(x) \\ =& N^4 \sum_{x\in V_N} N\nabla_i(\Delta(\chi u)\Delta v)(x) \\ &- N^4 \sum_{x\in V_N} (\Delta(\chi u))(x+e_i) N\nabla_i \Delta v(x). \end{aligned}$$

Now the first term is 0 due to the choice of the support of v, and the second - using the product rule on the discrete Laplacian - is equal to

$$\begin{aligned} &- N^4 \sum_{x\in V_N} \Delta u(x+e_i)\chi(x+e_i) N\nabla_i \Delta v(x) \\ &+ N^4 \sum_{x\in V_N} \sum_{\alpha:|\alpha|\le 2} \sum_{\substack{\beta:|\beta|\le 1 \\ |\alpha|+|\beta|=2}} k(\alpha,\beta)(\nabla^\alpha \chi)(x+e_i)(\nabla^\beta u)(x+e_i) N\nabla_i \Delta v(x) \end{aligned}$$

for suitable $k(\alpha, \beta) \in \mathbb{R}$. In the second term we use summation by parts and the regularity of χ to bound its absolute value from above by $c\|u\|_{H^2(V_N)}\|v\|_{H^2(V_N)}$. If we define the translation operator τ_i by $\tau_i(x) := x + e_i$, we can again use the product rule to rewrite the first term as

$$\begin{aligned}
-N^4 &\sum_{x\in V_N} \Delta u(x+e_i)\chi(x+e_i)N\nabla_i\Delta v(x)\\
&= - N^4 \sum_{x\in V_N} (\Delta u)(x+e_i)\Delta((\chi\circ\tau_i)N\nabla_i v)(x)\\
&\quad + N^4 \sum_{x\in V_N} (\Delta u)(x+e_i) \sum_{\alpha:|\alpha|\le 2} \sum_{\substack{\beta:|\beta|\le 1\\ |\alpha|+|\beta|=2}} k(\alpha,\beta)\nabla^\alpha\chi(x)\nabla^\beta N\nabla_i v(x).
\end{aligned}$$

Here, the first term is equal to

$$-N^4\mathcal{D}(u, \chi N\nabla_i(v\circ\tau^{-1})) = -N^4\langle g, \chi N\nabla_i(v\circ\tau^{-1})\rangle_{L_2(V_N)},$$

and the second is again bounded from above by $c\|u\|_{H^2(V_N)}\|v\|_{H^2(V_N)}$. $\square$

Proposition 2.5.3 *Let χ as in Lemma 2.5.2, and u the solution of (2.18). Then there exists $c > 0$ such that*

$$\|\chi u\|_{H^3(V_N)} \le cN^4\|g\|_{L_2(V_N)}.$$

Proof Note that

$$\begin{aligned}
|\langle g, N\chi\nabla_i\overline{v}\rangle_{L_2(V_N)}| \le&\|g\|_{L_2(V_N)}\|N\chi\nabla_i\overline{v}\|_{L_2(V_N)} \le c\|g\|_{L_2(V_N)}\|\overline{v}\|_{H^1(V_N)}\\
\le& c\|g\|_{L_2(V_N)}\|v\|_{H^2(V_N)}.
\end{aligned}$$

Thus if we set $v = N\nabla_i(\chi u)$ in Lemma 2.5.2 , we have, using Corollary A.1.6,

$$\begin{aligned}
\|N\nabla_i(\chi u)\|^2_{H^2(V_N)} \le& c_1 N^4\mathcal{D}(N\nabla_i(\chi u), N\nabla_i(\chi u))\\
\le& c_1\|N\nabla_i(\chi u)\|_{H^2(V_N)}(N^4\|g\|_{L_2(V_N)} + \|u\|_{H^2(V_N)}),
\end{aligned}$$

and so

$$\|N\nabla_i(\chi u)\|_{H^2(V_N)} \le c(N^4\|g\|_{L_2(V_N)} + \|u\|_{H^2(V_N)})$$

by Corollary A.1.6. The claim now follows from Corollary A.1.4 and Lemma 2.5.2. $\square$

Proposition 2.5.4 *Fix $3 \leq k \leq d/2+2$, and let χ be as in Lemma 2.5.2, and u the solution of (2.18). Then there exists $c > 0$ such that*

$$\|\chi u\|_{H^k(V_N)} \leq cN^4 \|g\|_{H^{k-3}(V_N)}.$$

Proof Apply the arguments of Lemma 2.5.2 and Proposition 2.5.3 repeatedly with $N\nabla_i u$, $N^2\nabla_i\nabla_j u$ etc. in the place of u, and similarly $N\nabla_i g, N^2\nabla_i\nabla_j g$ etc. in the place of g. □

Using the regularity of $\overline{G}_N$ and applying the Sobolev embedding Theorem now gives the desired result.

Corollary 2.5.5 *Let $d \geq 4$ and $0 < \delta < 1$. There exists a constant $c_d = c_d(\delta)$ such that for any $x \in V_N^\delta$,*

$$\sup_{y \in V_N^\delta} |G_N(x,y) - \overline{G}_N(x,y)| \leq c_d N^{4-d} \quad \text{as } N \to \infty.$$

Proof Chose $0 < \delta' < 1$ and $0 < \delta'' < \delta'/2$. We claim that exists a function $f : V_N \cup \partial_4 V_N$ which satisfies the following conditions: There is a constant $c = c(d,\delta) > 0$ sucht that

(a) $f(y) = \overline{G}_N(x,y)$ for all $y \in V_N \setminus V_N^{\delta''}$,

(b) $|\nabla^\alpha f(y)| \leq \frac{c}{N^{d+|\alpha|-4}}$ for all y in $V_N^{\delta'}$ and $|\alpha| \leq 4$

(c) $|\Delta^2 f(y)| \leq \frac{c}{N^d}$ for all $y \in V_N$.

are satisfied. The existence of such an f follows from Proposition 2.3.4 and the fact that $\Delta^2 \overline{G}_N(x,y) = 0$ if $y \in V_N \setminus V_N^\delta$. Simply set $f(y) = \overline{G}_N(x,y)$ if $\operatorname{dist}(y, \partial V_N) \leq \delta'' N$, and define f on $V_N^{\delta'}$ by setting it equal to any function satisfying (b). Then f has the required properties on $V_N^{\delta'} \cup \{y \in V_N : \operatorname{dist}(y,\partial V_N) \leq \delta'' N\}$, and we can continue it to all of V_N by interpolation, which is possible since the number of interpolation points is of order N^d.

Fix such an f satisfying (a), (b) and (c), and define $g := -\Delta^2 f$. Then $N^4\|g\|_{H^{k-3}(V_N)} \leq cN^{4-d/2}$ if $k < \delta'' N$, due to the choice of f. By Corollary 2.5.4, any solution of (2.18) satisfies $\|\chi u\|_{H^{d/2+1}(V_N)} \leq cN^{4-d/2}$, which implies $\|N^{d-4}\chi u\|_{H^{d/2+1}(V_N)} \leq cN^{d/2}$. But now Proposition A.2.1 yields $\sup_{y \in V_N} |N^{d-4}(\chi u)(y)| \leq c$. Since $\chi = 1$ on V_N^δ, this implies $\sup_{y \in V_N^\delta} |u(y)| \leq cN^{4-d}$. This proves the claim, as $G_N(x,y) - \overline{G}_N(x,y)$ solves (2.18) for any g like the one we constructed in this proof. □

We can now complete the proof of the variance estimates.

Proof of Proposition 2.1.1(b) and Proposition 2.1.2(b). We have seen in the previous section that

$$\mathrm{var}_N(\varphi_x) \leq \overline{G}_N(x,x) + O(N^{4-d}) \leq \gamma_d + O(N^{4-d}).$$

An the other hand, if $x \in V_N^\delta$,

$$\mathrm{var}_N(\varphi_x) \geq \overline{G}_N(x,x) + c(\delta)N^{4-d} \geq \gamma_d + c(\delta)N^{4-d}.$$

This proves Proposition 2.1.1. Proposition 2.1.2 follows in the same way, using Lemma 2.3.2. □

Remark 2.5.6 *In the proof of 2.3.3, the constant γ_d for $d \geq 5$, was identified as*

$$\gamma_d = \frac{d^2(d+2)\Gamma(d/2+1)}{\pi^{(d/2)}} + \Gamma(0,0)^2,$$

where $\Gamma(\cdot)$ denotes the Gamma function and $\Gamma(\cdot,\cdot)$ the harmonic Green's function.

Remark 2.5.7 *For Corollary 2.5.5 it is sufficient to have Proposition 2.5.4 satisfied with $k \leq d/2+1$. However, we will need $k \leq d/2+2$ in the four-dimensional situation later on for a stronger control on the covariances.*

Chapter 3
The supercritical case

3.1 Probability to stay positive

We follow a strategy introduced in [1], which was used in [12] for the case $k = K = 1$, $q_k = 1$. The idea is to use a conditioning argument on larger boxes than those of the proof of [20]. The main difficulty – when trying to follow the proof for the free field – arises when considering the expectations of φ_x conditioned on the boundary of a box of side-length L. While in the case of the harmonic crystal, we know by the random walk representation, that on Ω_N^+ the conditional expectations are nonnegative, in our more general case they can be strictly negative. We overcome this difficulty by estimating the proportion of conditional expectations that are of order $-N^\lambda$, $\lambda \in \mathbb{N}$. Then we prove that this proportion is negligible if we let N tend to infinity.

Proof of Theorem 1.3.1, the upper bound. Fix a natural number $L > K+1$ such that $L-K$ is even, and let $\overline{\Lambda} = (L, L, ..., L) + L\mathbb{Z}^d$. For $x \in \overline{\Lambda}$ denote by $\partial B(x) = \{y \in \mathbb{Z}^d : \max_{i=1,...,d} |x_i - y_i| \in [\frac{L-K}{2}, \frac{L+K}{2}]\}$ the boundary of the box $B(x) := \{y \in \mathbb{Z}^d : \max_{i=1,...,d} |x_i - y_i| < \frac{L-K}{2}\}$. Let $\tilde{\Lambda} = \{x \in \overline{\Lambda} : \partial B(x) \subset V_N\}$ and $\Lambda = \cup_{x\in\tilde{\Lambda}} \partial B(x)$.

Since $J(x, y) = 0$ for $|x - y| > K$, the field $\{\varphi_x\}_{x\in\tilde{\Lambda}}$ is Markovian, in the sense that $P(\,\cdot \,|\, \mathcal{F}_{B(x)^c}) = P(\,\cdot\,|\,\mathcal{F}_{\partial B(x)})$ for all $x \in \tilde{\Lambda}$, and thus (see [11], Proposition 13.13), under $P(\,\cdot\,|\,\mathcal{F}_{B(x)^c})$, the φ_x, $x \in \tilde{\Lambda}$, are independent normally distributed random variables. For the mean and the variance we write

$$m_x = E\left(\varphi_x \,|\, \mathcal{F}_{B(x)^c}\right) \quad \text{and} \quad G_L = \operatorname{var}\left(\varphi_x \,|\, \mathcal{F}_{B(x)^c}\right)$$

respectively. Note that $\lim_{L\to\infty} G_L = G$ (see [11], Section 13.1). For any subset A of $\mathbb{Z}^d$ let Ω_A^+ denote the event $\{\varphi_x \geq 0 \quad \forall x \in A\}$. Because of the independence we have

$$P\left(\Omega_N^+\right) \le P\left(\Omega_\Lambda^+ \cap \Omega_{\tilde\Lambda}^+\right) \le E\left[\prod_{x\in\tilde\Lambda} P(\varphi_x \ge 0 \,|\, \mathcal{F}_{\partial B(x)}) \cdot 1_{\Omega_\Lambda^+}\right]. \quad (3.20)$$

As in [12], we use a decomposition of V on a larger scale: Let $\theta > 0$, $r \in \mathbb{R}^d$ and set $A_r = r + [0,\theta)^d$, and $I = \{r \in \theta\mathbb{Z}^d : A_r \subset V, \partial A_r \cap \partial V = \emptyset\}$. Set $\tilde B_r = NA_r \cap \tilde\Lambda$, the box containing the centres of the smaller boxes $B(x)$, with $x \in NA_r$. Note that $B := |\tilde B_r| = O(N^d)$.

Let $0 < \delta < 1$ and $0 < \gamma < 1$. For $\kappa > 0$, let $a_N = \sqrt{4k(G-\kappa)\log N}$ and consider the following events:

$$E_{\delta,\kappa} = \left\{\varphi : \text{there is}\, r \in I \,\text{such that}\, \left|\{x \in \tilde B_r : m_x \le a_N\}\right| \ge \delta B\right\},$$

$$E_\delta^{-\lambda} = \Big\{\varphi : \text{there is}\, r \in I \,\text{such that}$$

$$\left|\{x \in \tilde B_r : m_x \le -N^\lambda\}\right| \ge \frac{\delta}{N^{2k+(2-\gamma)\lambda-\gamma}} B\Big\},$$

where λ is a nonnegative integer. Note that $N^{-2k-(2-\gamma)\lambda+\gamma}\delta B < 1$ for $\lambda \ge \lambda_{\max} = \lfloor (d-2k+\gamma)/(2-\gamma)\rfloor + 1$ (where $\lfloor\cdot\rfloor$ denotes the integer part). For these λs, $E_\delta^{-\lambda}$ is $\{\varphi : \text{there is}\, r \in I \,\text{such that}\; \{x \in \tilde B_r : m_x \le -N^\lambda\} \ne \emptyset\}$, and these $E_\delta^{-\lambda}$s are all contained in $E_\delta^{-\lambda_{\max}}$. Set

$$F_\delta = \bigcup_{\lambda=0}^{\lambda_{\max}} E_\delta^{-\lambda}.$$

The estimate (3.20) now gives

$$P(\Omega_N^+) \le E\left[\prod_{x\in\tilde\Lambda} P(\varphi_x \ge 0 \,\big|\, \mathcal{F}_{\partial B(x)}) \cdot 1_{\Omega_\Lambda^+ \cap F}\right]$$
$$+ E\left[\prod_{x\in\tilde\Lambda} P(\varphi_x \ge 0 \,\big|\, \mathcal{F}_{\partial B(x)}) \cdot 1_{\Omega_\Lambda^+ \cap F^c}\right],$$

where $F = E_{\delta,\kappa} \cup F_\delta$. The following lemma shows that we can estimate $\prod_{x\in\tilde\Lambda} P(\varphi_x \ge 0 \,|\, \mathcal{F}_{\partial B(x)})$ uniformly on F :

Lemma 3.1.1 *Let $0 < \gamma < 1$. The following hold:*

(a) For L large enough, there exist a constant $c_1 > 0$ such that

$$E\left[\prod_{x\in\tilde{\Lambda}} P(\varphi_x \geq 0 \,|\, \mathcal{F}_{\partial B(x)}) \cdot 1_{\Omega_\Lambda^+ \cap E_{\delta,\kappa}}\right] \leq \exp\left(-c_1 N^{d-2k+\gamma}\right) \quad (3.21)$$

(b) For N large enough, there exists a constant c_2 such that

$$E\left[\prod_{x\in\tilde{\Lambda}} P(\varphi_x \geq 0 \,|\, \mathcal{F}_{\partial B(x)}) \cdot 1_{\Omega_\Lambda^+ \cap F_\delta}\right] \leq \exp\left(-c_2 N^{d-2k+\gamma}\right). \quad (3.22)$$

Both constants depend on L, θ and δ but not on N.

Proof In both cases, we use standard estimates on the centred Gaussian variables $m_x - \varphi_x$ under $P(\,\cdot\,|\mathcal{F}_{\partial B(x)})$.

(a) Since $G_L \longrightarrow G$, we have that, for L large enough, $4k(G-\kappa)/2G_L \leq 2k - \gamma$. We therefore get on $E_{\delta,\kappa}$

$$\begin{aligned}
\prod_{x\in\tilde{\Lambda}} P(\varphi_x \geq 0 \,|\, \mathcal{F}_{\partial B(x)}) &\leq P\left(\varphi_0 - m_0 \leq a_N \,\middle|\, \mathcal{F}_{\partial B(x)}\right)^{\delta B} \\
&\leq \left(1 - \frac{\sqrt{G_L}}{a_N} \exp\left(-\frac{a_N^2}{2G_L}\right)\right)^{\delta B} \\
&\leq \exp\left(-c_1 N^{d-2k+\gamma}\right).
\end{aligned}$$

(b) On F_δ we have for some constants $C > 0, c_2 > 0$, and for N large enough

$$\begin{aligned}
\prod_{x\in\tilde{\Lambda}} P(\varphi_x \geq 0 \,|\, \mathcal{F}_{\partial B(x)}) &\leq \sum_{\lambda=0}^{\infty} P(\varphi_0 - m_0 \geq N^\lambda \,|\, \mathcal{F}_{\partial B(x)})^{\delta N^{-2k-(2-\gamma)\lambda+\gamma} B} \\
&\leq \sum_{\lambda=0}^{\infty} \left(\exp\left(-\frac{N^{2\lambda}}{2G_L}\right)\right)^{\delta N^{-2k-(2-\gamma)\lambda+\gamma} B} \\
&\leq \sum_{\lambda=0}^{\infty} \exp\left(-C N^{d-2k+\gamma}\right)^{N^{\gamma\lambda}} \\
&\leq \exp\left(-c_2 N^{d-2k+\gamma}\right).
\end{aligned}$$

$\square$

Therefore we only need to consider F^c, where we can easily bound $\sum_{x\in\tilde{B}_r} m_x$. Write

$$\sum_{x\in\tilde{B}_r} m_x = \sum_{x:m_x>a_N} m_x + \sum_{x:-1<m_x\le a_N} m_x + \sum_{\lambda=0}^{\lambda_{\max}} \sum_{x:-N^{\lambda+1}<m_x\le N^{\lambda}} m_x$$

and bound the three parts separately: On $E^c_{\delta,\kappa}$, at least $(1-\delta)$ of the m_x are at height at least a_N, so for the first part we get

$$\sum_{m_x>a_N} m_x \ge (1-\delta)B\, a_N\,. \tag{3.23}$$

The second term can be estimated easily by writing

$$\sum_{-1<m_x\le a_N} m_x \ge -B\,. \tag{3.24}$$

Finally, since on F^c_δ there is

$$\begin{aligned} \left|\{x\in\tilde{B}_r : -N^{\lambda+1} < m_x \le -N^{\lambda}\}\right| \le \Big|\{x\in\tilde{B}_r : m_x \\ \le -N^{\lambda}\}\Big| \le \frac{\delta}{N^{2k+(2-\gamma)\lambda-\gamma}} B, \end{aligned}$$

we get

$$\begin{aligned} \sum_{\lambda=0}^{\lambda_{\max}} \left[\sum_{-N^{\lambda+1}<m_x\le N^{\lambda}} m_x\right] &\ge -\sum_{\lambda=0}^{\lambda_{\max}} B\cdot\delta\cdot N^{-2k-(2-\gamma)\lambda+\gamma}\cdot N^{\lambda+1} \\ &= -B\cdot\delta\cdot N^{-2k+\gamma+1}\sum_{\lambda=0}^{\lambda_{\max}} N^{-(1-\gamma)\lambda} \\ &\ge -c\cdot B\cdot N^{-2k+\gamma+1}. \end{aligned} \tag{3.25}$$

The three estimates (3.23), (3.24), (3.25) together give

$$\frac{1}{B}\sum_{x\in\tilde{B}_r} m_x \ge (1-\delta)a_N + O(1) \tag{3.26}$$

on F^c. Let $f_r \geq 0$ ($r \in I$). Then (3.26) implies

$$\begin{aligned} P(\Omega_N^+ \cap F^c) &\leq P\left(\sum_{r\in I} f_r \frac{1}{B} \sum_{x\in\tilde{B}_r} m_x > (1-\delta)a_N \sum_{r\in I} f_r + O(1)\right) \\ &\leq \exp\left(\frac{-(1-\delta)^2 a_N^2 (\sum_{r\in I} f_r)^2 + O(\sqrt{\log N})}{2\mathrm{var}(\sum_{r\in I} f_r \frac{1}{B}\sum_{x\in\tilde{B}_r} m_x)}\right) \quad (3.27) \end{aligned}$$

Now we can conclude the proof of the upper bound as in [4]. Since m_x is the conditional expectation $E(\varphi_x \,|\, \mathcal{F}_{\partial B(x)}) = E(\varphi_x \,|\, \mathcal{F}_\Lambda)$, we have by Jensen's inequality

$$\mathrm{var}\left(\sum_{r\in I} f_r \frac{1}{B} \sum_{x\in\tilde{B}_r} m_x\right) \leq \mathrm{var}\left(\sum_{r\in I} f_r \frac{1}{B} \sum_{x\in\tilde{B}_r} \varphi_x\right).$$

Define $f_\theta : \mathbb{R}^d \to \mathbb{R}$ by $f_\theta(t) = \sum_{r\in I} f_r 1_{A_r}(t)$. One easily sees that

$$\sum_{r\in I} f_r \frac{1}{B} \sum_{x\in\tilde{B}_r} \varphi_x = \frac{1}{B}\sum_{x\in\tilde{\Lambda}} f_\theta\left(\frac{x}{N}\right)\varphi_x \quad \text{and} \quad \sum_{r\in I} f_r = \frac{1}{B}\sum_{x\in\tilde{\Lambda}} f_\theta\left(\frac{x}{N}\right),$$

and consequently

$$\mathrm{var}\left(\frac{1}{B}\sum_{r\in I} f_r \sum_{x\in\tilde{B}_r} \varphi_x\right) = \frac{1}{B^2}\sum_{x,y\in\tilde{\Lambda}} f_\theta\left(\frac{x}{N}\right) f_\theta\left(\frac{y}{N}\right) G(x,y).$$

Thus we obtain, using Lemma 3.1.1 and (3.27),

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{N^{d-2k}\log N} \log P(\Omega_N^+) \leq \limsup_{N\to\infty} \frac{1}{N^{d-2k}\log N} \log P(\Omega_\Lambda^+ \cap F^c) \\ &\leq \limsup_{N\to\infty} \frac{1}{N^{d-2k}\log N} \frac{-(1-\delta)^2 4k(G-\kappa)\log N (\sum_{r\in I} f_r)^2}{2\mathrm{var}(\sum_{r\in I} f_r \frac{1}{B}\sum_{x\in\tilde{B}_r} m_x)} \\ &= -(1-\delta)^2 2k(G-\kappa) \limsup_{N\to\infty} \frac{1}{N^{d-2k}} \cdot \frac{(\sum_{x\in\tilde{\Lambda}} f_\theta(\frac{x}{N}))^2}{\sum_{x,y\in\tilde{\Lambda}} f_\theta(\frac{x}{N}) f_\theta(\frac{y}{N}) G(x,y)}. \end{aligned}$$

As in [4], the proof is now concluded by applying Proposition B.3.2, taking the supremum over all possible f_θ and letting $\kappa \to 0$ and $\delta \to 0$.

3.2 Entropic repulsion

To prove Theorem 1.3.2, there are two directions to show. The first was proved in Theorem 2.2 of [20]: For any $\varepsilon > 0, \eta > 0$, and $z \in V_N$, such that $V_{N,\varepsilon}(z) \subset V_N$,

$$\lim_{N\to\infty} P\left(\frac{\overline{\varphi}_{N,\varepsilon}(z)}{\sqrt{\log N}} \le \sqrt{4kG} - \eta \,\middle|\, \Omega_N^+\right) = 0\,. \tag{3.28}$$

We will now use Theorem 1.3.1 to show the other bound:

Proposition 3.2.1 *For any $\varepsilon > 0, \eta > 0$ and $z \in V_N$, with $V_{N,\varepsilon}(z) \subset V_N$*

$$\lim_{N\to\infty} P\left(\frac{\overline{\varphi}_{N,\varepsilon}(z)}{\sqrt{\log N}} \ge \sqrt{4kG} + \eta \,\middle|\, \Omega_N^+\right) = 0\,. \tag{3.29}$$

The proof for the lattice free field in [4] uses the FKG-inequality for the conditional measure, which does not hold in our case. Similarly to Section 3.1, we can handle this problem by carefully estimating the probability that, on Ω_N^+, the local sample mean of the field is higher than $\sqrt{4kG \cdot \log N}$. This is done by comparing $\overline{\varphi}_{N,\varepsilon}(z)$ with the average of the conditional expectations m_x.

Proof First, let $z = 0$, set $\overline{\varphi}_{N,\varepsilon} := \overline{\varphi}_{N,\varepsilon}(z)$, and $V_{N,\varepsilon} := V_{N,\varepsilon}(0)$. Fix L as in Section 3.1 and recall the definition of the subgrid $\overline{\Lambda}$, the boxes $B(x)$ and their $K-$boundary $\partial B(x)$. In this section, $\tilde{\Lambda}$ denotes the set $\{x \in \overline{\Lambda} : \partial B(x) \subset V_{N,\varepsilon}\}$, and $\Lambda = \cup_{x\in\tilde{\Lambda}} \partial B(x)$. For $r \in \mathbb{R}^d$ and $0 < \theta < 1$ let A_r be defined as in Section 3.1, and set $I = \{r \in \theta\mathbb{Z}^d : A_r \subset V_\varepsilon\}, \partial A_r \cap \partial V_\varepsilon = \emptyset\}$, where $V_\varepsilon = [-\varepsilon, \varepsilon]^d$. Set $B_r = NA_r$, and $\tilde{B}_r = B_r \cap \tilde{\Lambda}$. As before, set $m_x := E(\varphi_x \,|\, \mathcal{F}_{\partial B(x)})$ for $x \in \tilde{B}_r$.

We want to estimate

$$P\big(\overline{\varphi}_{N,\varepsilon} \ge (\sqrt{4kG} + \eta)\sqrt{\log N} \,\big|\, \Omega_N^+\big)$$
$$= \frac{1}{P(\Omega_N^+)}\, P\left(\{\overline{\varphi}_{N,\varepsilon} \ge (\sqrt{4kG} + \eta)\sqrt{\log N}\} \cap \Omega_N^+\right)\,.$$

Recall F from Section 3, fix $t > 0$ and set $D_t := \{\varphi : \text{there is}\, r \in I\, \text{such that}\, \frac{1}{|\tilde{B}_r|}\sum_{x\in\tilde{B}_r}(\varphi_x - m_x) < -t\}$. Then we can write

$$\begin{aligned}
P\big(\{\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG}+\eta)\sqrt{\log N}\} \cap \Omega_N^+\big)&\\
&= P\left(\{\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG}+\eta)\sqrt{\log N}\} \cap \Omega_N^+ \cap F\right)\\
&\quad + P\left(\{\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG}+\eta)\sqrt{\log N}\} \cap \Omega_N^+ \cap F^c \cap D_t\right)\\
&\quad + P\left(\{\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG}+\eta)\sqrt{\log N}\} \cap \Omega_N^+ \cap F^c \cap D_t^c\right) .
\end{aligned}$$

We have seen in the last section that the first term is negligible compared to $P(\Omega_N^+)$. For the second part, recall that conditioned on $\mathcal{F}_{\partial B(x)}$, the $\varphi_x - m_x$, $x \in \tilde{B}_r$, are independent centred Gaussian variables with variance G_L. Thus for the variance of the average we get

$$\begin{aligned}
\operatorname{var}\left(\frac{1}{|\tilde{B}_r|}\sum_{x\in\tilde{B}_r}(\varphi_x - m_x)\,\middle|\, \mathcal{F}_\Lambda\right) &= \frac{1}{|\tilde{B}_r|^2}\sum_{x\in\tilde{B}_r}\operatorname{var}\left(\varphi_x - m_x \,\middle|\, \mathcal{F}_{\partial B(x)}\right)\\
&= \frac{1}{|\tilde{B}_r|}\cdot G_L .
\end{aligned}$$

We can therefore find constants $c_1 > 0$, and $c_2 = c_2(\theta) > 0$ such that

$$\begin{aligned}
P\big(D_t \cap F^c \cap \Omega_N^+\big)&\\
&\leq c_2 E\left(P\left(\frac{1}{|\tilde{B}_r|}\sum_{x\in\tilde{B}_r}(\varphi_x - m_x) < -t \,\middle|\, \mathcal{F}_\Lambda\right)\cdot 1_{F^c\cap\Omega_N^+}\right)\\
&\leq c_2 \exp\left(\frac{-t^2\cdot c_1 N^d}{2G_L}\right),
\end{aligned} \tag{3.30}$$

which is also negligible compared with $P(\Omega_N^+)$. Therefore we only need to estimate

$$\limsup_{N\to\infty}\frac{1}{P(\Omega_N^+)}P\left(\{\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG}+\eta)\sqrt{\log N}\} \cap \Omega_N^+ \cap F^c \cap D_t^c\right) .$$

For this purpose we bound $\frac{1}{|\tilde{B}_r|}\sum_{x\in\tilde{B}_r}\varphi_x$ from below on $\Omega_N^+\cap F^c\cap D_t^c$. Write

$$\frac{1}{|\tilde{B}_r|}\sum_{x\in\tilde{B}_r}\varphi_x=\frac{1}{|\tilde{B}_r|}\sum_{x\in\tilde{B}_r}(\varphi_x-m_x)+\frac{1}{|\tilde{B}_r|}\sum_{x\in\tilde{B}_r}m_x \tag{3.31}$$

and recall from the last section, that on $\Omega_N^+\cap F^c$

$$\frac{1}{|\tilde{B}_r|}\sum_{x\in\tilde{B}_r}m_x\geq(1-\delta)a_N+O(1) \tag{3.32}$$

for any $\delta>0$ and $\kappa>0$. This implies that on $\Omega_N^+\cap F^c\cap D_t^c$, we have

$$\frac{1}{|\tilde{B}_r|}\sum_{x\in\tilde{B}_r}\varphi_x\geq(1-\delta)a_N+O(1)\,.$$

Since we can repeat this argument with any shift of the subgrid Λ, and average over all shifts, we conclude that on $\Omega_N^+\cap F^c\cap D_t^c$

$$\frac{1}{|B_r|}\sum_{x\in B_r}\varphi_x\geq(1-\delta)a_N+O(1)\,. \tag{3.33}$$

From now on we will abbreviate $\frac{1}{|B_r|}\sum_{x\in B_r}\varphi_x$ by $\overline{\varphi}_r$. For $\kappa'>0$, set

$$C_{\kappa'}:=\{\varphi:\text{ there exists } r_0 \text{ such that } \overline{\varphi}_{r_0}\geq(\sqrt{4k(G-\kappa)}+\kappa')\sqrt{\log N}\}.$$

It follows from (3.33) that, on $\Omega_N^+\cap F^c\cap D_t^c$, for every $\eta>0$ we can find $\kappa'>0$ and $r_0\in I$ such that, for $N\to\infty$,

$$P\left(\overline{\varphi}_{N,\varepsilon}\geq(\sqrt{4kG}+\eta)\sqrt{\log N}\right)\leq P\left(\{\overline{\varphi}_r\geq(1-\delta)a_N\ \forall r\in I\}\cap C_{\kappa'}\right)\,.$$

Let $f_r>0, r\in I$.

$$\begin{aligned}
P\Big(\overline{\varphi}_r\geq&(1-\delta)a_N\quad\forall r\in I,\ \overline{\varphi}_{r_0}\geq(\sqrt{4k(G-\kappa)}+\kappa')\sqrt{\log N}\Big)\\
&\leq P\left(\sum_{r\in I}f_r\overline{\varphi}_r\geq(1-\delta)a_N\cdot\sum_{r\in I}f_r+\kappa' f_{r_0}\sqrt{\log N}\right)\\
&\leq\exp\left(\frac{-\left((1-\delta)a_N\sum_{r\in I}f_r+\kappa' f_{r_0}\sqrt{\log N}\right)^2}{2\mathrm{var}\left(\sum_{r\in I}f_r\overline{\varphi}_r\right)}\right)\,.
\end{aligned}$$

Defining f_θ as in the last section, we have

$$\sum_{r\in I} f_r = \frac{1}{|B_r|} \sum_{x\in V_{N,\varepsilon}} f_\theta\left(\frac{x}{N}\right)$$

and

$$\begin{aligned}
\operatorname{var}\left(\sum_{r\in I} f_r \overline{\varphi}_r\right) &= \frac{1}{|B_r|^2} \sum_{r\in I}\sum_{s\in I} f_r f_s \sum_{x\in B_r}\sum_{y\in B_s} G(x,y) \\
&= \frac{1}{|B_r|^2} \sum_{x,y\in V_{N,\varepsilon}} f_\theta\left(\frac{x}{N}\right) f_\theta\left(\frac{x}{N}\right) G(x,y) \\
&= O(N^{-d+2k})\,.
\end{aligned}$$

Similarly to the end of Section 3.1, we can then optimise over f_θ, use Proposition B.3.2, and let κ and δ tend to 0. Then we see that there is a constant $c > 0$, such that

$$\begin{aligned}
\limsup_{N\to\infty} \frac{1}{N^{d-2k}\log N} \log P\big(\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG}+\eta)\sqrt{\log N}\big)& \\
\leq -2k\cdot G\cdot \mathcal{C}_k(V) - c\,.&
\end{aligned}$$

Now we apply Theorem 1.3.1, and obtain

$$\limsup_{N\to\infty} \frac{1}{N^{d-2k}\log N} \log P\left(\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG}+\eta)\sqrt{\log N} \;\middle|\; \Omega_N^+\right) \leq -c\,,$$

which proves the claim in the case $z = 0$. The case of an arbitrary z is obtained by repeating the same arguments on a shifted grid. $\square$

Theorem 1.3.2 now follows immediately from (3.28) and Proposition 3.2.1. This proves the height estimate.

Chapter 4
The critical case

4.3 Strategy of the proof

The proofs of the entropic repulsion results for the two-dimensional lattice free field in [3] were motivated by the observation that, due to the logarithmic covariances, the field strongly resembles a hierarchical (or ultrametric) one. In particular, the behaviour of the maximum is obtained by proving that the “non-hierarchical” part of the field is negligible in this context and by controlling the errors. The main ingredients in this approach are

- logarithmic variances
- concentration property
- convergence of the correlations.

For the four-dimensionale membrane model, we have the same logarithmic variances (Proposition 2.1.2). Concerning the correlations, the results of Appendix B.4 are sufficient for our purposes. In order to approximate the field by a hierarchical one, the concentration property of Lemma 4.3.4 below will be crucial. To obtain this, we introduce the following quantity:

$$a(x,y) := \sum_{k=0}^{\infty}(k+1)\big(\mathbb{P}^x(X_k = x) - \mathbb{P}^x(X_k = y)\big).$$

Lemma 4.3.2 below shows that this is finite for any pair $x, y \in \mathbb{Z}^d$.

Remark 4.3.1 *Note that $a(0,0) = 0$, and that $a(x,y) = a(0, y-x)$. It is easily verified by direct computation that $\Delta^2 a(x,y) = \delta(x,y)$. Thus we call $a(x,y)$ is a discrete version of the fundamental solution for the* *bilaplacian, in analogy to the usual terminology for partial differential equations. For the continuous biharmonic operator, the fundamental solution in* $d = 4$ *is given by* $\frac{1}{8\pi^2}\log|x-y|$ *(see e.g. [21]).*

The local central limit theorem ([17], Theorem 1.2.1) allows us to compute $a(x,y)$. The constant we obtain is different from the continuous case due to our choice of the normalisation of the discrete Laplacian.

Lemma 4.3.2 *Let* $d = 4$. *There exists a constant* K *such that for all* $y \neq 0$ *and for all* $0 < \alpha < 2$,

$$a(0,y) = \frac{8}{\pi^2}\log|y| + K + o(|y|^{-\alpha}). \tag{4.34}$$

Proof First, note that $a(0,y) = \sum_k k\left(\mathbb{P}^0(X_k = 0) - \mathbb{P}^0(X_k = y)\right) + \Gamma(0,0) - \Gamma(0,y)$. Recall that $\Gamma(0,y) \leq O(|y|^{-2})$, and $\Gamma(0,0)$ is a constant. We use the notation and results of Section 2.2. Let us first assume that y is even. Then

$$\sum_{k=0}^{\infty} k\big(\mathbb{P}^0(X_k = 0) - \mathbb{P}^0(X_k = y)\big)$$
$$= \sum_{k=1}^{\infty} 2k\left(\mathbb{P}^0(X_{2k} = 0) - \mathbb{P}^0(X_{2k} = y)\right)$$

and

$$\sum_{k=1}^{\infty} 2k\big(\mathbb{P}^0(X_{2k} = 0) - \mathbb{P}^0(X_{2k} = y)\big)$$
$$= \sum_{k=1}^{\infty} 2k\left(\overline{p}(2k,0) - \overline{p}(2k,y) + E(2k,0) - E(2k,y)\right).$$

We first consider the remainder term. From the local CLT with error bounds (Lemma 2.2.1, compare [17], Theorem 1.2.1) we know

$$|E(k,y)| \leq O(k^{-3}) \quad \text{and} \quad |E(k,y)| \leq |y|^{-2}O(k^{-2}),$$

and consequently

$$\begin{aligned}\sum_{k=1}^{\infty} 2kE(2k,y) \leq& \sum_{k\leq|y|^2/2} 2kE(2k,y) + \sum_{k>|y|^2/2} 2kE(2k,y)\\ \leq&|y|^2 \sum_{k\leq|y|^2/2} E(2k,y) + \sum_{k>|y|^2/2} 2kO((2k)^{-3})\\ \leq&|y|^2 \sum_{k\leq|y|^2/2} E(2k,y) + O(|y|^{-2}).\end{aligned}$$

But from Lemma 1.5.2 of [17] we know that $\sum_{k=0}^{\infty} E(k,y) = o(|y|^{-\alpha})$ for any $\alpha < 4$ as $|y| \to \infty$.
Now consider the other term. By definition,

$$\sum_{k=1}^{\infty} 2k\big(\overline{p}(2k,0) - \overline{p}(2k,y)\big) = \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k}\Big(1 - \exp\big(-|y|^2/k\big)\Big).$$

Now use exactly the same steps as in the proof of Theorem 1.6.2 of [17] to show that there is a constant $\tilde{K}$ such that

$$\frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k}\Big(1 - \exp\big(-|y|^2/k\big)\Big) = \frac{4}{\pi^2}\Big(\log|y|^2 + \tilde{K} + O(|y|^{-2})\Big).$$

Let $K = \Gamma(0,0) + \frac{4}{\pi^2}\tilde{K} + \sum_{k=1}^{\infty} 2kE(2k,0)$. This proves case where y is even. If y is odd,

$$\begin{aligned}&\sum_{k=0}^{\infty} k\big(\mathbb{P}^0(X_k = 0) - \mathbb{P}^0(X_k = y)\big)\\ &\qquad=\sum_{k=1}^{\infty} 2k\,\big(\mathbb{P}^0(X_{2k} = 0) - \mathbb{P}^0(X_{2k+1} = y)\big) - \Gamma(0,y)\\ &\qquad=\frac{1}{2d}\sum_{v:|y-v|=1}\sum_{k=1}^{\infty} 2k\,\big(\mathbb{P}^0(X_{2k} = 0) - \mathbb{P}^0(X_{2k} = v)\big) - \Gamma(0,y).\end{aligned}$$

Of course all these v are even, so we obtain, since $\frac{1}{2d}\sum_{v:|y-v|=1}\log|v|^2 = \log|y|^2 + O(|y|^{-2})$,

$$a(0,y) = \frac{4}{2d\pi^2}\sum_{v:|y-v|=1}\log|v|^2 + K + o(|y|^{-\alpha}) = \frac{8}{\pi^2}\log|y| + K + O(|y|^{-\alpha}),$$

where $\alpha < 2$ and K is the same as before. $\square$

Lemma 4.3.3 *Let $d = 4$ and let $x \in V_N^\delta$. There exists $c > 0$ such that*

$$\sup_{y \in V_N^\delta} |\nabla(G_N(x,y) - \overline{G}_N(x,y))| \leq \frac{c}{N}.$$

Proof This follows from Proposition 2.5.4 and Corollary A.2.2. $\square$

Lemma 4.3.4 *Let $0 < n < N$, let $A_N \subset \mathbb{Z}^4$ be a box of side-length N, $A_n \subset A_N$ a box of side-length n with the same center $x_B \in \mathbb{Z}^4$ as A_N. Let $0 < \varepsilon < 1/2$. There exists $c > 0$ such that for all $x \in A_n$ with $|x - x_B| \leq \varepsilon n$,*

$$\operatorname{var}\left(E(\varphi_x|\mathcal{F}_{\partial_2 A_n}) - E(\varphi_{x_B}|\mathcal{F}_{\partial_2 A_n})\middle|\mathcal{F}_{\partial_2 A_N}\right) \leq c\varepsilon.$$

Proof Note that for any two subsets $E \subset F$ of $\mathbb{Z}^4$,

$$\begin{aligned}\operatorname{var}(\varphi_x|\mathcal{F}_{F^c}) &= \operatorname{var}(\varphi_x|\mathcal{F}_{E^c}) + \operatorname{var}\left(E(\varphi_x|\mathcal{F}_{E^c})|\mathcal{F}_{F^c}\right) \\ &\geq \operatorname{var}(\varphi_x|\mathcal{F}_{E^c}).\end{aligned} \tag{4.35}$$

Let $B_n := B_n(x_B) = \{z \in \mathbb{Z}^d : |x_B - z| < n\}$ the ball of radius n and centre x_B. Note $B_n \subset A_n$, and so

$$\begin{aligned}&\operatorname{var}\Big(E(\varphi_x - \varphi_{x_B}|\mathcal{F}_{\partial_2 A_n})\Big|\mathcal{F}_{\partial_2 A_N}\Big) \\ &\qquad= \operatorname{var}(\varphi_x - \varphi_{x_B}|\mathcal{F}_{A_N^c}) - \operatorname{var}(\varphi_x - \varphi_{x_B}|\mathcal{F}_{A_n^c}) \\ &\qquad\leq \lim_{N\to\infty}\left(\operatorname{var}(\varphi_x - \varphi_{x_B}|\mathcal{F}_{A_N^c}) - \operatorname{var}(\varphi_x - \varphi_{x_B}|\mathcal{F}_{B_n^c})\right) \\ &\qquad= \lim_{N\to\infty}(G_N(x,x) - 2G_N(x,x_B) + G_N(x_B,x_B) \\ &\qquad\quad - G_{B_n}(x,x) + 2G_{B_n}(x,x_B) - G_{B_n}(x_B,x_B)).\end{aligned}$$

(Of course we don't know if the limit exists, but otherwise the rhs is equal to $+\infty$.) Now, $G_N = \overline{G}_N + H_N$. From Corollary 4.3.3 we know that $|H_N(y,z) - H_N(y,z+e_i)| \leq cN^{-1}$, and since $|x - x_B| \leq \varepsilon n$, we need at most $4\varepsilon n$ steps to get from x_B to x. Thus $|H_N(y,x) - H_N(y,x_B)| \leq \varepsilon n \cdot cN^{-1}$ if $y \in \{x, x_B\}$, and so

$$\begin{aligned}&\lim_{N\to\infty}(H_N(x,x) - 2H_N(x,x_B) + H_N(x_B,x_B) \\ &\qquad - H_{B_n}(x,x) + 2H_{B_n}(x,x_B) - H_{B_n}(x_B,x_B)) \\ &\quad\leq \lim_{N\to\infty} \varepsilon n \cdot cN^{-1} + \varepsilon n \cdot cn^{-1} \leq c\varepsilon.\end{aligned}$$

We are therefore left with estimating the terms involving $\overline{G}_N$ and $\overline{G}_{B_n}$. We have

$$
\begin{aligned}
&\overline{G}_N(x,x) - 2\overline{G}_N(x,x_B) + \overline{G}_N(x_B,x_B)\\
&\qquad\qquad\qquad - \overline{G}_{B_n}(x,x) + 2\overline{G}_{B_n}(x,x_B) - \overline{G}_{B_n}(x_B,x_B)\\
&= \sum_{k=0}^{\infty}(k+1)\Big[\mathbb{P}^x(X_k = x, \tau_{B_n} \le k \le \tau_N)\\
&\quad - \mathbb{P}^x(X_k = x_B, \tau_{B_n} \le k \le \tau_N) + \mathbb{P}^{x_B}(X_k = x_B, \tau_{B_n} \le k \le \tau_N)\\
&\quad - \mathbb{P}^{x_B}(X_k = x, \tau_{B_n} \le k \le \tau_N)\Big]
\end{aligned}
$$

Hence, using the above monotonicity (4.35), we are done if we show

$$
\begin{aligned}
T(x,x_B) := &\sum_{k=0}^{\infty}(k+1)\Big[\mathbb{P}^x(X_k = x, k \ge \tau_{B_n}) - \mathbb{P}^x(X_k = x_B, k \ge \tau_{B_n})\\
&+ \mathbb{P}^{x_B}(X_k = x_B, k \ge \tau_{B_n}) - \mathbb{P}^{x_B}(X_k = x, k \ge \tau_{B_n})\Big] \le c\varepsilon.
\end{aligned}
$$

By the Markov property,

$$
\mathbb{P}^x(X_k = x, k \ge \tau_{B_n}) = \sum_{z \in \partial B_n}\sum_{m=0}^{\infty} \mathbb{P}^z(X_{k-m} = x)\mathbb{P}^{x_B}(\tau_{B_n} = m, X_{\tau_{B_n}} = z)
$$

and similarly for $\mathbb{P}^x(X_k = x_B, k \ge \tau_{B_n})$ etc. This implies, shifting the summation index,

$$
\begin{aligned}
T(x,x_B) = &\sum_{k=0}^{\infty}\sum_{z \in \partial B_n}\sum_{m=0}^{\infty}(k+m+1)\Big[\mathbb{P}^x(\tau_{B_n} = m, X_{\tau_{B_n}} = z)\\
&- \mathbb{P}^{x_B}(\tau_{B_n} = m, X_{\tau_{B_n}} = z)\Big]\Big[\mathbb{P}^z(X_k = x)) - \mathbb{P}^z(X_k = x_B)\Big]\\
= &T_1 + T_2,
\end{aligned}
$$

if we define

$$
\begin{aligned}
T_1 := &\sum_{k=0}^{\infty}\sum_{z \in \partial B_n}(k+1)\big(\mathbb{P}^z(X_k = x_B) - \mathbb{P}^z(X_k = x)\big)\\
&\times \sum_{m=0}^{\infty}\big(\mathbb{P}^x(\tau_{B_n} = m, X_{\tau_{B_n}} = z) - \mathbb{P}^{x_B}(\tau_{B_n} = m, X_{\tau_{B_n}} = z)\big),
\end{aligned}
$$

and

$$T_2 := \sum_{k=0}^{\infty} \sum_{z\in\partial B_n} \big(\mathbb{P}^z(X_k = x) - \mathbb{P}^z(X_k = x_B)\big) \\ \times \sum_{m=0}^{\infty} m\big(\mathbb{P}^x(\tau_{B_n} = m, X_{\tau_{B_n}} = z) - \mathbb{P}^{x_B}(\tau_{B_n} = m, X_{\tau_{B_n}} = z)\big).$$

Due to Lemma 4.3.2, for x, x_B as above, $\sup_{z\in\partial B_n} |a(z, x_B) - a(z, x)| \leq c\varepsilon$, which implies

$$T_1 = \sum_{z\in\partial B_n} \big(\mathbb{P}^x(X_{\tau_{B_n}} = z) - \mathbb{P}^{x_B}(X_{\tau_{B_n}} = z)\big)\big(a(z, x_B) - a(z, x)\big) \leq c\varepsilon.$$

For the second term, note that

$$\sum_{z\in\partial B_n} \sum_{m=0}^{\infty} m\big(\mathbb{P}^x(\tau_{B_n} = m, X_{\tau_{B_n}} = z) - \mathbb{P}^{x_B}(\tau_{B_n} = m, X_{\tau_{B_n}} = z)\big) \\ = \mathbb{E}^x(\tau_{B_n}) - \mathbb{E}^{x_B}(\tau_{B_n}),$$

and from [17], Equation 1.21, we know that

$$n^2 - |y - x_B|^2 \leq \mathbb{E}^y(\tau_{B_n}) \leq (n+1)^2 - |y - x_B|^2$$

for all $y \in B_n$. Therefore $|\mathbb{E}^x(\tau_{B_n}) - \mathbb{E}^{x_B}(\tau_{B_n})| \leq \varepsilon^2 n^2 + 2n + 1$. On the other hand, by construction, $|z - x_B| \geq n$ and $|z - x| \geq (1-\varepsilon)n$, and hence

$$\sup_{z\in\partial B_n} \sum_{k=0}^{\infty} \mathbb{P}^z(X_k = x) = \sup_{z\in\partial B_n} \Gamma(z, x) \leq \frac{c}{(1-\varepsilon)^2 n^2},$$

and likewise for $\sum_{k=0}^{\infty} \mathbb{P}^z(X_k = x_B)$. This implies $|T_2| \leq c\varepsilon$, if n is large enough, which concludes the proof.

□

4.4 Maximum of the field

In this section, we prove Theorem 1.3.3, using the strategy of [3] and [8]. Let $\alpha \in (1/2, 1)$. We cover V_N^δ with boxes of side-length N^α as in [3]:

Let $x_0 \in V_N$, and let

$$M_\alpha := \{x_0 + i(N^\alpha + 2) : i = (i_1, ..., i_4) \in \mathbb{N}^4 \text{ s.th. } x_0 + i(N^\alpha + 2) \subset V_N\}.$$

We consider the set of boxes B with midpoint in M_α and side-length N^α. We will always assume that N^α is an odd integer, which is no restriction as $N \to \infty$. By construction, the boundaries between two boxes have thickness 2 (on the lattice), which is the range of interactions of Δ^2. Let Π_α denote the set of such boxes which are contained in V_N^δ, and let $\Lambda_\alpha := \bigcup_{B \in \Pi_\alpha} \partial_2 B$ be the set of all boundaries of boxes in Π_α. We denote by $\mathcal{F}_\alpha$ the sigma-algebra generated by the φ_x with $x \in \Lambda_\alpha$. Conditional on $\mathcal{F}_\alpha$, what happens inside different boxes is independent.
Now fix $K \in \mathbb{N}$. Set $\alpha_i := \alpha(1 - \frac{i-1}{K}), 1 \leq i \leq K + 1$. We define the following sets of boxes: First, let $\Gamma_{\alpha_1} := \Pi_{\alpha_1}$. Then $\Gamma_{\alpha_i}, i \geq 2$, is defined recursively: For $B \in \Gamma_{\alpha_{i-1}}$, let $\Gamma_{B,\alpha_i} := \{B' \in \Pi_{\alpha_i} : B' \subset B/2\}$, and $\Gamma_{\alpha_i} := \bigcup_{B \in \Gamma_{\alpha_{i-1}}} \Gamma_{B,\alpha_i}$. For $B \in \Pi_\alpha$, we denote the midpoint of B by x_B. Let

$$\varphi_B := E_N(\varphi_{x_B} | \mathcal{F}_{\partial_2 B}) = E_N(\varphi_{x_B} | \mathcal{F}_\alpha).$$

If $B \in \Pi_{\alpha_i}$ and $B' \in \Pi_{\alpha_j}$, with $\alpha_i \leq \alpha_j$ such that $x_B = x_{B'}$, by (4.35) and Proposition 2.1.2 we see that

$$\begin{aligned} \operatorname{var}(\varphi_B | \mathcal{F}_{\alpha_j}) =& \operatorname{var}(\varphi_{x_B} | \mathcal{F}_{\alpha_j}) - \operatorname{var}(\varphi_{x_B} | \mathcal{F}_{\alpha_i}) \\ =& \gamma(\alpha_j - \alpha_i) \log N + O(1). \end{aligned} \tag{4.36}$$

Note that by (1.10), there exist coefficients $h(z) \in \mathbb{R}$ such that

$$\varphi_B = \sum_{z \in \partial_2 B} h(z) \varphi_z.$$

Unlike in the case of the lattice free field however, the $h(z)$ need not lie between 0 and 1 (in fact, one can see that there are both positive and negative coefficients, and they need not be bounded in N). Some arguments in the proof need to be adapted to this fact, in particular, comparing φ_B and φ_{x_B} requires some work, for which we use Gaussian tail estimates. For the sake of readability, we give a complete proof, including also those parts that are practically identical to [3] or [8]. Note that one direction is easy to prove:

Proof of Theorem 1.3.3(a): Using Proposition 2.1.2(a), we obtain

$$P_N\left(\sup_{x\in V_N}\varphi_x \geq 2\sqrt{2\gamma}\log N\right) \leq |V_N| \sup_{x\in V_N} P_N\left(\varphi_x \geq 2\sqrt{2\gamma}\log N\right)$$
$$\leq N^4 \frac{\sqrt{\gamma\log N + c}}{\sqrt{2\pi}2\sqrt{2\gamma}\log N}\exp\left(-\frac{(2\sqrt{2\gamma}\log N)^2}{2\gamma\log N + O(1)}\right)$$

which tends to zero as $N\to\infty$. □

The second part is obtained from the following more general result (compare [8]):

Theorem 4.4.1 *Let* $0<\delta<1/2$, *and let* $0<\lambda<1$. *For all* $\varepsilon>0$. *There exists* $c=c(\delta)>0$ *such that*

$$P_N\left(|\{x\in V_N^\delta : \varphi_x \geq 2\sqrt{2\gamma}\lambda\log N\}| \leq N^{4(1-\lambda^2)-\varepsilon}\right) \leq \exp(-c(\log N)^2).$$

Proof of Theorem 1.3.3 (b): Chose in Theorem 4.4.1 λ sufficiently close to 1, such that $2\sqrt{2\gamma}\lambda \geq (2\sqrt{2\gamma}-\eta)$ and $4\lambda^2 > 4-\varepsilon$ are both satisfied. □

To prove Theorem 4.4.1,we start on level $\alpha=\alpha_1$ of the box structure introduced before, and show that on this level, a sufficiently high number of the $\varphi_B, B\in\Gamma_\alpha$, are positive:

Lemma 4.4.2 *Let* $1/2<\delta<1$ *and* $\alpha\in(1/2,1)$. *There exist positive constants* κ, a *depending on* α *and* δ, *such that*

$$P_N\left(|\{B\in\Gamma_\alpha : \varphi_B\geq 0\}| \leq N^\kappa\right) \leq \exp\left(-a(\log N)^2\right).$$

Proof Set $\alpha'=(1+\alpha)/2$, which implies $\alpha'>\alpha$. We consider the event

$$A := \left\{\sharp\{B\in\Pi_{\alpha'} : \varphi_B \geq \frac{-(1-\alpha')\sqrt{2\gamma}\log N}{2}\} \geq N^{1-\alpha'}\right\}.$$

The lemma will be proven showing that the following two estimates hold:

$$P_N\left(A\cap\{\sharp\{B\in\Gamma_\alpha : \varphi_B\geq 0\}\leq N^\kappa\}\right) \leq \exp\left(-c(\log N)^2\right) \tag{4.37}$$

for some $c>0$, and

$$P_N(A^c) \leq \exp\left(-c(\log N)^2\right). \tag{4.38}$$

Obviously, these two estimates prove the lemma. We start with the second estimate. Let us split the event A^c into

$$\begin{aligned} P_N(A^c) \leq & P_N\big(A^c \cap \{\max_{B\in\Pi_{\alpha'}} \varphi_B \leq (\log N)^2\}\big) \\ & + P_N\big(\max_{B\in\Pi_{\alpha'}} \varphi_B > (\log N)^2\big) \end{aligned} \tag{4.39}$$

and bound the two terms. First, notice that for any $0 < \rho < 1$ we have

$$\begin{aligned} P_N\left(\max_{x\in V_N} \varphi_x > (1-\rho)(\log N)^2\right) \leq & N^4 \max_{x\in V_N} P_N\left(\varphi_x > (1-\rho)(\log N)^2\right) \\ \leq & N^4 \exp\left(-\frac{(1-\rho)^2(\log N)^4}{2\gamma\log N + C}\right) \\ \leq & \exp\left(-c(\log N)^3\right). \end{aligned}$$

Now we get

$$\begin{aligned} & P_N\big(\{\max_{B\in\Pi_{\alpha'}} \varphi_B > (\log N)^2\} \cap \{\max_{x\in V_N} \varphi_x \leq (1-\rho)(\log N)^2\}\big) \\ & \quad \leq P_N\left(\{\max_{B\in\Pi_{\alpha'}} \varphi_B > (\log N)^2\} \cap \{\max_{B\in\Pi_{\alpha'}} \varphi_{x_B} \leq (1-\rho)(\log N)^2\}\right) \\ & \quad \leq |\Pi_{\alpha'}| \max_{B\in\Pi_{\alpha'}} P_N\left(\{\varphi_B > (\log N)^2\} \cap \{\varphi_{x_B} \leq (1-\rho)(\log N)^2\}\right) \\ & \quad \leq cN^4 E_N\left(P_N\left(\varphi_{x_{B_0}} \leq (1-\rho)(\log N)^2 | \mathcal{F}_{\partial B_0}\right) 1_{\{\varphi_{B_0} > (\log N)^2\}}\right). \end{aligned}$$

for some fixed $B_0 \in \Pi_{\alpha'}$. But on $\{\varphi_{B_0} > (\log N)^2\}$, we have

$$\begin{aligned} & P_N\big(\varphi_{x_{B_0}} \leq (1-\rho)(\log N)^2 | \mathcal{F}_{\partial B_0}\big) \\ & \quad \leq P_N\left(\varphi_{x_{B_0}} - \varphi_{B_0} \leq -\rho(\log N)^2 | \mathcal{F}_{\partial B_0}\right) \leq \exp\left(-c(\log N)^3\right). \end{aligned}$$

This gives the required bound on the second term in (4.39). To bound the first term, note that on $A^c \cap \{\max_{B\in\Pi_{\alpha'}} \varphi_B \leq (\log N)^2\}$ we have

$$\begin{aligned} & \frac{1}{|\Pi_{\alpha'}|} \sum_{B\in\Pi_{\alpha'}} \varphi_B \\ & \quad \leq \frac{-(1-\alpha')\sqrt{2\gamma}\log N}{2} + \frac{N^{1-\alpha'}}{|\Pi_{\alpha'}|}\left(\frac{(1-\alpha')\sqrt{2\gamma}\log N}{2} + (\log N)^2\right). \end{aligned}$$

Since $|\Pi_{\alpha'}| = O(N^{4(1-\alpha')})$, we get

$$\frac{1}{|\Pi_{\alpha'}|}\sum_{B\in\Pi_{\alpha'}}\varphi_B \le \frac{-(1-\alpha')\sqrt{2\gamma}\log N}{3},$$

and this implies with Lemma B.4.1

$$\begin{aligned}
&P_N\Big(A^c \cap \{\max_{B\in\Pi_{\alpha'}}\varphi_B \le (\log N)^2\}\Big)\\
&\qquad\le P_N\left(\frac{1}{|\Pi_{\alpha'}|}\sum_{B\in\Pi_{\alpha'}}\varphi_B \le \frac{-(1-\alpha')\sqrt{2\gamma}\log N}{3}\right)\\
&\qquad\le \exp\left(\frac{-(1-\alpha')^2\gamma(\log N)^2}{9\ \mathrm{var}\left(\frac{1}{|\Pi_{\alpha'}|}\sum_{B\in\Pi_{\alpha'}}\varphi_B\right)}\right)\\
&\qquad\le \exp(-c(\log N)^2).
\end{aligned}$$

This proves (4.38). For the proof of (4.37), we consider only the set of boxes in Π_α which have the same centre as some box of $\Pi_{\alpha'}$: Let

$$\Pi_{\alpha,\alpha'} := \{B\in\Pi_\alpha : \exists B'\in\Pi_{\alpha'} \text{ with } x_B = x_{B'}\}.$$

We have

$$\begin{aligned}
&P_N\big(A\cap\{|\{B\in\Gamma_\alpha : \varphi_B \ge 0\}| \le N^\kappa\}\big)\\
&\qquad\le P_N\left(A\cap\{|\{B\in\Pi_{\alpha,\alpha'} : \varphi_B\ge 0\}|\le N^\kappa\}\right)\\
&\qquad\le E_N\left(P_N\left(|\{B\in\Pi_{\alpha,\alpha'} : \varphi_B\ge 0\}|\le N^\kappa|\mathcal{F}_{\alpha'}\right)1_A\right).
\end{aligned}$$

We know that on A there exist at least $N^{1-\alpha'}$ boxes $B'\in\Pi_{\alpha'}$ where there is $\varphi_{B'} \ge -(1-\alpha')\sqrt{2\gamma}\log N/2$. Choose $N^{1-\alpha'}$ of them and call them $B'_1, ..., B'_{N^{1-\alpha'}}$. Let $B_i\in\Pi_{\alpha,\alpha'}$ be the box with centre $x_{B_i} = x_{B'_i}$. Set $\zeta_i = \varphi_{B_i} - \varphi_{B'_i}$. By construction, we have: $\varphi_{B'_i} = E_N(\varphi_{x_{B'_i}}|\mathcal{F}_{\alpha'}) = E_N\left(E_N(\varphi_{x_{B_i}}|\mathcal{F}_\alpha)|\mathcal{F}_{\alpha'}\right) = E_N(\varphi_{B_i}|\mathcal{F}_{\alpha'})$. Therefore we know:

- The ζ_i are centred Gaussian random variables under $P_N(\cdot|\mathcal{F}_{\alpha'})$
- By (4.36), $\mathrm{var}(\zeta_i) = \mathrm{var}_{B'_i}(\varphi_{B_i}) = \gamma(1-\alpha')\log N + O(1)$.

Then for $\kappa < 1-\alpha'$,

$$P_N\big(|\{B \in \Pi_{\alpha,\alpha'} : \varphi_B \geq 0\}| \leq N^\kappa | \mathcal{F}_{\alpha'}\big)$$
$$\leq P_N\left(\sum_{i=1}^{N^{1-\alpha'}} 1_{\{\zeta_i \geq \frac{1-\alpha'}{2}\sqrt{2\gamma}\log N\}} \leq N^\kappa\right),$$

and

$$P_N\left(\zeta_i \geq \frac{1-\alpha'}{2}\sqrt{2\gamma}\log N\right) \geq \exp\left(\frac{-(1-\alpha')\log N}{4}\right) = N^{-(1-\alpha')/4}.$$

If we choose now $\kappa = (1-\alpha')/2$ and set $\theta_i = 1_{\{\zeta_i \geq (1-\alpha')\sqrt{2\gamma}\log N/2\}}$, on A we have $\sum_{i=1}^{N^{1-\alpha'}} \theta_i \leq N^{(1-\alpha')/2}$ and $E\theta_i \geq N^{-(1-\alpha')/4}$. This implies

$$\Big|\sum_{i=1}^{N^{1-\alpha'}} (\theta_i - E\theta_i)\Big| \geq |N^{(1-\alpha')/2} - N^{1-\alpha'} \cdot N^{(1-\alpha')/4}| \geq \frac{N^{3(1-\alpha')/4}}{2},$$

from which we conclude, using Lemma 11 of [3],

$$P_N\Big(\sum_{i=1}^{N^{1-\alpha'}} 1_{\{\zeta_i \geq \frac{1-\alpha'}{2}\sqrt{2\gamma}\log N\}} \leq N^{(1-\alpha')/2}\Big)$$
$$\leq P_N\left(\Big|\sum_{i=1}^{N^{1-\alpha}} (\theta_i - E\theta_i)\Big| \geq \frac{N^{3(1-\alpha')/4}}{2}\right)$$
$$\leq \exp\left(-\frac{N^{3(1-\alpha')/2}}{4(2N^{1-\alpha'} + N^{3(1-\alpha')/4})/3}\right)$$
$$\leq \exp(-cN^{(1-\alpha')/2}).$$

This is more than we need to prove (4.37). □

Proof of Theorem 4.4.1: Fix $\alpha \in (1/2, 1)$. From the previous lemma we know that we can find some $\kappa = \kappa(\alpha) > 0$, such that we can assume that at least N^κ of the $\varphi_B, B \in \Pi_\alpha$, are positive. We use the notation of the previous section, and define, for $1 \leq k \leq K+1$, and $\varepsilon > 0$, the event

$$\begin{aligned}A_k &:= A_k(\varepsilon,\alpha,K)\\ &= \bigcup_{B'\in\Gamma_{\alpha_k}}\bigcup_{B\in\Gamma_{B',\alpha_{k+1}}}\{|\varphi_{B'}-E_N(\varphi_B|\mathcal{F}_{\alpha_k})|\geq \varepsilon\lambda\alpha 2\sqrt{2\gamma}\frac{1}{K}(1-\frac{1}{K})\log N\}.\end{aligned}$$

By Lemma 4.3.4, $\mathrm{var}(\varphi_{B'}-E(\varphi_B|\mathcal{F}_{\alpha_k})|\mathcal{F}_{\alpha_{k+1}})\leq c$, and we can bound

$$\begin{aligned}P(A_k)\leq&|\Gamma_{\alpha_k}||\Gamma_{B',\alpha_{k+1}}|\exp\left(-\frac{\varepsilon^2\lambda^2\alpha^2 8\gamma\frac{1}{K^2}(1-1/K)^2(\log N)^2}{2c}\right)\\ \leq&\exp(-c(\log N)^2).\end{aligned}\tag{4.40}$$

We will later choose $K\geq\varepsilon\lambda$, such that c is independent of ε and λ.

On $\cap_k A_k^c$, we can apply the tree-argument of [3]. For $k\leq K$ we denote by $\underline{B}^{(k)}$ a sequence of k boxes $B_1\supset B_2\supset ...\supset B_k$, where $B_i\in\Gamma_{\alpha_i}, 1\leq i\leq k$. Set

$$D_k := \left\{\underline{B}^{(k)} : \varphi_{B_i}\geq(\alpha-\alpha_i)\lambda 2\sqrt{2\gamma}(1-1/K)\log N,\quad 1\leq i\leq k\right\}.$$

We show that if on the $k-$th scale, there are many such sequences, so there will be on the $k+1$st scale. Let $n_k := N^{\kappa+4\alpha(k-1)\frac{1}{K}(1-\lambda)^2}$, where κ is the same constant as in Lemma 4.4.2, and define

$$C_k := \left\{|D_k|\geq n_k\right\}.$$

Assuming we are on C_k, choose n_k sequences $\underline{B}_j^{(k)}=\{B_{j,1},B_{j,2},...,B_{j,k}\}$, $1\leq j\leq n_k$ in D_k. Note that $B_{j,k}\neq B_{i,k}$ if $i\neq j$, since otherwise the sequences would coincide. Set

$$\zeta_j := \frac{1}{|\Gamma_{B_{j,k},\alpha_{k+1}}|}\sum_{B\in\Gamma_{B_{j,k},\alpha_{k+1}}} 1_{\{\varphi_B-\varphi_{B_{j,k}}\geq\lambda\alpha 2\sqrt{2\gamma}\frac{1}{K}(1-\frac{1}{K})\log N\}}$$

Note that $|\Gamma_{B_{j,k},\alpha_{k+1}}|=(N^{\alpha/K}/2)^4$, and therefore

$$C_k\cap C_{k+1}^c\subset C_k\cap\left\{\sum_{j=1}^{n_k}\zeta_j\leq n_{k+1}\cdot\frac{16}{N^{4\alpha/K}}\right\}.$$

If we set

$$\tilde{\zeta}_j := \frac{1}{|\Gamma_{B_{j,k},\alpha_{k+1}}|} \sum_{B\in\Gamma_{B_{j,k},\alpha_{k+1}}} 1_{\{\varphi_B - E(\varphi_B|\mathcal{F}_{\alpha_k}) \geq (1+\varepsilon)\lambda\alpha 2\sqrt{2\gamma}\frac{1}{K}(1-\frac{1}{K})\log N\}},$$

we have $\zeta_j \geq \tilde{\zeta}_j$ on A_k^c, and therefore

$$P_N(C_k \cap C_{k+1}^c \cap A_k^c) \leq P_N\left(\sum_{j=1}^{n_k} \tilde{\zeta}_j \leq n_{k+1} \cdot \frac{16}{N^{4\alpha/K}}\right).$$

To bound this probability, we need some large deviation estimates on the binomial variables $\sum_{j=1}^{n_k} \tilde{\zeta}_j$. Note that the $\varphi_B - E_N(\varphi_B|\mathcal{F}_{\alpha_k})$ are centred Gaussian variables with variance

$$\mathrm{var}(\varphi_B|\mathcal{F}_{\alpha_k}) \geq \frac{\alpha}{K}\gamma\log N + c.$$

Therefore

$$\begin{aligned}
&E_N(\tilde{\zeta}_j|\mathcal{F}_{\alpha_k})\\
&\geq \inf_B P_N\left(\varphi_B - E_N(\varphi_B|\mathcal{F}_{\alpha_k}) \geq (1+\varepsilon)\lambda\alpha 2\sqrt{2\gamma}\frac{1}{K}(1-\frac{1}{K})\log N \Big| \mathcal{F}_{\alpha_k}\right)\\
&\geq \exp\left(-\frac{(1+\varepsilon)^2\lambda^2\alpha^2 8\gamma(1/K^2)(1-1/K)^2(\log N)^2}{2\alpha(1/K)\gamma\log N}\right)\\
&= N^{-4\frac{\alpha}{K}\lambda^2(1-\frac{1}{K})^2(1+\varepsilon)^2}.
\end{aligned}$$

Thus on $C_k \cap A_k^c$,

$$\begin{aligned}
C_{k+1}^c \subset &\left\{\sum_{j=1}^{n_k}(\tilde{\zeta}_j - E(\tilde{\zeta}_j|\mathcal{F}_{\alpha_k})) \leq n_{k+1}\frac{16}{N^{4\frac{\alpha}{K}}} - n_k N^{-4\frac{\alpha}{K}\lambda^2(1-\frac{1}{K})^2(1+\varepsilon)^2}\right\}\\
&\subset \left\{\Big|\sum_{j=1}^{N^\kappa}(\tilde{\zeta}_j - E(\tilde{\zeta}_j|\mathcal{F}_{\alpha_k}))\Big| \geq \frac{1}{2}N^{\kappa - \frac{4\alpha}{K}\lambda^2(1-1/K)^2(1+\varepsilon)^2}\right\},
\end{aligned}$$

if, for the last line, ε is chosen such that $(1-1/K)(1+\varepsilon) < 1$, making the second term dominate. Then Lemma 11 of [3] yields on $C_k \cap A_k^c$,

$$\begin{aligned} P_N(C_{k+1}^c|\mathcal{F}_{\alpha_k}) \leq& 2\exp\left(-\frac{N^{2\kappa-8\lambda^2\frac{\alpha}{K}(1-\frac{1}{K})^2(1+\varepsilon)^2}}{2N^\kappa+(2/3)N^{\kappa-4\lambda^2\frac{\alpha}{K}(1-\frac{1}{K})^2(1+\varepsilon)^2}}\right) \\ \leq& \exp\left(-N^{\kappa-8\lambda^2\frac{\alpha}{K}(1-\frac{1}{K})^2(1+\varepsilon)^2}\right). \end{aligned} \tag{4.41}$$

If we choose K large enough, such that $\kappa-\frac{8\alpha}{K}>0$, this implies

$$\begin{aligned} P_N(C_K^c) \leq& P_N(C_1^c)+\sum_{k=2}^{K}(P_N(C_k^c\cap C_{k-1}\cap A_{k-1}^c)+P_N(A_{k-1})) \\ =& P_N(C_1^c)+\sum_{k=2}^{K}E_N(P_N(C_k^c|F_{\alpha_k})1_{C_{k-1}\cap A_{k-1}^c})+P_N(A_{k-1})) \\ \leq& \exp(-c_1(\log N)^2)+K\exp\left(-N^{\kappa-8\lambda^2\frac{\alpha}{K}(1-\frac{1}{K})^2(1+\varepsilon)^2}\right) \\ &+\exp(-c_2(\log N)^2) \\ \leq& \exp(-c(\log N)^2). \end{aligned}$$

Let now $\mathcal{H}_N(a):=\{x\in V_N^\delta:\varphi_x\geq 2\sqrt{2\gamma}a\log N\}$. We consider the event

$$L_K=L_K(\alpha,\lambda):=\{|\mathcal{H}_N(\lambda(\alpha-\alpha_{K-1}))|\leq n_{K-1}\}.$$

Note that

$$P_N(L_K)\leq P_N(|\{B\in\Pi_{\alpha_K}:\varphi_{x_B}\geq 2\sqrt{2\gamma}\lambda(\alpha-\alpha_{K-1})\log N\}|\leq n_{K-1}).$$

This implies

$$\begin{aligned} P(L_K\cap C_K)\leq E_N\big(P_N\big(&|\{B\in\Pi_{\alpha_K}:\varphi_{x_B}\geq 2\sqrt{2\gamma}\lambda(\alpha-\alpha_{K-1})\log N\}| \\ &\leq n_{K-1}\big|\mathcal{F}_{\alpha_K}\big)1_{C_K}\big). \end{aligned}$$

On $C_K\cap L_K$ we have at least n_K boxes $B\in\Pi_{\alpha_K}$ with $\varphi_B\geq 2\sqrt{2\gamma}\lambda(\alpha-\alpha_K)\log N$, and only for at most n_{K-1} of them we have $\varphi_{x_B}\geq 2\sqrt{2\gamma}\lambda(\alpha-\alpha_K)\log N$. Thus for at least n_K-n_{K-1} boxes, $\varphi_{x_B}-\varphi_B\leq 2\sqrt{2\gamma}\lambda(\alpha_K-\alpha_{K-1})\log N$. Now we use the fact that, conditional on $\mathcal{F}_{\alpha_K}$, the $\varphi_{x_B}-\varphi_B$ are independent centered Gaussian with variance equal to $\gamma\alpha_K \log N$, and that $\alpha_K - \alpha_{K-1} = -\frac{\alpha}{K} < 0$, and $n_{K-1} = n_K N^{-\frac{4\alpha}{K}(1-\lambda^2)}$ to obtain

$$\begin{aligned}
&P_N\big(|\{B \in \Pi_{\alpha_K} : \varphi_{x_B} \geq 2\sqrt{2\gamma}\lambda(\alpha - \alpha_{K-1})\log N\}| \leq n_{K-1}\big|\mathcal{F}_{\alpha_K}\big)\\
&\leq P_N\big(|\{B : \varphi_{x_B} - \varphi_B \leq -\frac{\alpha}{K}2\sqrt{2\gamma}\lambda \log N\}| \geq n_K - n_{K-1}\big|\mathcal{F}_{\alpha_K}\big)\\
&\leq P_N\big(\varphi_{x_B} - \varphi_B \leq -\frac{\alpha}{K}2\sqrt{2\gamma}\lambda(1-1/K)\log N|\mathcal{F}_{\alpha_K}\big)^{(1-N^{-\frac{4\alpha}{K}(1-\lambda^2)})n_K}\\
&\leq \exp\big(-4\lambda^2\frac{\alpha}{K}(1-1/K)^2(\log N)(1-N^{-\frac{4\alpha}{K}(1-\lambda^2)})n_K\big)\\
&\leq \exp(-c(\log N)^2).
\end{aligned} \tag{4.42}$$

To complete the proof, we get, using $\alpha - \alpha_{K-1} = \alpha(1-\frac{2}{K})$,

$$\begin{aligned}
P_N\big(|\mathcal{H}_N(\lambda\alpha(1-\frac{2}{K})| \leq n_{K-1}\big) &\leq P_N\big(L_K \cap C_K) + P_N(C_K^c\big)\\
&\leq \exp(-c(\log N)^2).
\end{aligned} \tag{4.43}$$

We can now choose K large enough an α close to 1, such that with (4.43)

$$\begin{aligned}
P_N\big(|\{x \in V_N^\delta :&\varphi_x \geq 2\sqrt{2\gamma}\lambda \log N\}| \leq N^{4(1-\lambda^2)-\varepsilon}\big)\\
&\leq P_N\big(|\mathcal{H}_N(\lambda\alpha(1-\frac{2}{K})| \leq n_{K-1}\big)\\
&\leq \exp(-c(\log N)^2).
\end{aligned}$$

□

4.5 Probability to stay positive

Proof of Theorem 1.3.4, the lower bound. First, note that by a density argument, $\mathcal{C}_V^2(D) = \inf\{\frac{1}{2}\int_V |\Delta h|^2 dx : \ h \in C_0^\infty(V), h \geq 1 \text{ a.e. on } D\}$, where $C_0^\infty(V)$ denotes the infinitely often differentiable functions on V which vanish at ∂V. Choose a function $f \in C_0^\infty(V), f \geq 0, f = 1$ on D, and a number $a > 2\sqrt{2\gamma}$. Set $\tilde{\varphi}_x := \varphi_x + a \log N f(\frac{x}{N})$. Then $\{\tilde{\varphi}_x\}_{x \in V_N}$ is a Gaussian family with covariances $G_N(x,y), x, y \in$

V_N, and expectation $a \log N f(\frac{x}{N})$. Denote the law of this family by P_N^a, and let $f_N(x) := f(x/N)$. The relative entropy of P_N^a with respect to P_N is defined as $H_N(P_N^a|P_N) := E_N^a\left(\log \frac{dP_N^a}{dP_N}\right)$. Note that

$$\frac{dP_N^a}{dP_N}(\varphi) = \exp\Big[\frac{1}{2}(\langle\varphi, G_N^{-1}\varphi\rangle_{V_N} \\ - \langle\varphi - a\log N f_N, G_N^{-1}(\varphi - a\log N f_N)\rangle_{V_N})\Big],$$

where $\langle\cdot,\cdot\rangle_{V_N}$ denotes the L_2-scalar product on V_N, and therefore

$$E_N^a\left(\log\frac{dP_N^a}{dP_N}\right) = \frac{a^2}{2}(\log N)^2\langle\Delta_N f_N, \Delta_N f_N\rangle_{V_N},$$

from which we conclude

$$\lim_{N\to\infty}\frac{1}{(\log N)^2}H_N(P_N^a|P_N) = \frac{a^2}{2}\|\Delta f\|^2_{L^2(V)}.$$

Moreover,

$$P_N^a\left((\Omega_N^+)^c\right) \leq \sum_{x\in D_N} P_N^a(\varphi_x < 0) = \sum_{x\in D_N} P_N(\varphi_x < -a\log N) \\ \leq N^4\exp\left(\frac{-a^2(\log N)^2}{2\gamma\log N}\right) = o(1)$$

as $N\to\infty$. Using the entropy inequality (see for example [12], Appendix B.3)we have

$$\log\frac{P_N(\Omega_N^+)}{P_N^a(\Omega_N^+)} \geq -\frac{H_N(P_N^a|P_N) + e^{-1}}{P_N^a(\Omega_N^+)}$$

and hence

$$\liminf_{N\to\infty}\frac{1}{(\log N)^2}\log P_N(\Omega_N^+) \geq -\frac{a^2}{2}\|\Delta f\|_{L^2(V)}$$

for any choice of a and f as above. Optimizing over a and f gives the lower bound. □

Proof of the upper bound. Fix $\beta > 0$. For $K \in \mathbb{N}, \alpha \in (1/2, 1)$ define

$$E_{K,\beta,\alpha} := \left\{\sharp\{B \in \Pi_\alpha : B \subset D_N, \varphi_B \leq (2\sqrt{2\gamma} - \beta)\log N\} \leq K\right\}$$

the event that we have few boxes $B \in \Pi_\alpha$ with $\varphi_B \le (2\sqrt{2\gamma} - \beta)\log N$. We will now show that the probability that Ω_N^+ occurs on $E^c_{K,\beta,\alpha}$ is small. If $\eta > 0, \varepsilon \in (0, 1/2), \alpha \in (0,1)$ let

$$A := \bigcup_{B \in \Pi_\alpha} \bigcup_{x \in B^{(\varepsilon)}} \{|\varphi_B - E_N(\varphi_x|\mathcal{F}_\alpha)| \ge \eta \log N\},$$

where $B^{(\varepsilon)}$ is the set of points $x \in B$ which are contained inside a box of side-length εN^α and centre x_B. We split

$$P_N(E^c_{K,\beta,\alpha} \cap \Omega^+_{D_N}) \le E_N\left(P_N(E^c_{K,\beta,\alpha} \cap \Omega^+_{D_N})|\mathcal{F}_\alpha)1_{A^c}\right) + P_N(A).$$

But by Lemma 4.3.4, we find

$$P_N(A) \le N^4 \exp\left(-\frac{\eta^2(\log N)^2}{c\varepsilon}\right) \le \exp\left(-\frac{c'\eta^2(\log N)^2}{\varepsilon}\right).$$

We can choose ε arbitrarily small, our choice will be such that $\frac{c'\eta^2}{\varepsilon} \ge 8\gamma\mathcal{C}_V^2(D)+1$. Fix $B \in \Pi_\alpha$, and set $B^{(\varepsilon)} := \{x \in B : \mathrm{dist}(x, \partial B) \ge \varepsilon N^\alpha\}$. The idea is to apply Theorem 1.3.3 to the field $(\varphi_x - E_N(\varphi_x|\mathcal{F}_\alpha))_{x\in B}$ conditional on $\mathcal{F}_\alpha$. We get

$$\begin{aligned}
P_N\big(&\sup_{x\in B^{(\varepsilon)}} (\varphi_x - E_N(\varphi_x|\mathcal{F}_\alpha)) \le (2\sqrt{2\gamma} - \beta)\log N|\mathcal{F}_\alpha\big) \\
&\le P_N\big(\sup_{x\in B^{(\varepsilon)}} (\varphi_x - E(\varphi_x|\mathcal{F}_\alpha)) \le (2\sqrt{2\gamma} - \beta/2)\log N^\alpha|\mathcal{F}_\alpha\big) \\
&\le \exp(-c(\log N)^2),
\end{aligned}$$

where $c = c(\varepsilon, \beta)$ if $\alpha \in (\alpha_0(\beta), 1)$ for some $\alpha_0(\beta) > 0$. Therefore on $A^c \cap \{\varphi : \varphi_B \le (2\sqrt{2\gamma} - \beta)\log N\}$ we have if $\eta \le \beta/2$,

$$\begin{aligned}
P_N\big(&\inf_{x\in B} \varphi_x \ge 0|\mathcal{F}_\alpha\big) \\
&\le P_N\big(\inf_{x\in B^{(\varepsilon)}} (\varphi_x - E_N(\varphi_x|\mathcal{F}_\alpha)) \ge -(2\sqrt{2\gamma} - \beta/2)\log N|\mathcal{F}_\alpha\big) \\
&\le \exp(-c(\log N)^2)
\end{aligned}$$

if $\alpha \geq a_0(\beta)$. This implies

$$\begin{aligned}
&P_N(E^c_{K,\beta,\alpha} \cap \Omega^+_N) \\
&\leq \begin{pmatrix} N^{4-4\alpha} \\ K \end{pmatrix} \big(\exp(-c(\log N)^2)\big)^K + \exp\big(-(8\gamma\mathcal{C}^2_V(D)+1)(\log N)^2\big) \\
&\quad \leq \exp\big((4-4\alpha)K\log N - cK(\log N)^2\big) \\
&\qquad\qquad + \exp\big(-(8\gamma\mathcal{C}^2_V(D)+1)(\log N)^2\big) \\
&\quad \leq \exp\big(-(8\gamma\mathcal{C}^2_V(D)+1)(\log N)^2\big)
\end{aligned}$$

if we choose K large enough such that $cK/2 \geq 8\gamma\mathcal{C}^2_V(D)+1$.
This means we now only need to consider $E_{K,\beta,\alpha} \cap \Omega^+_{D_N}$. In this case, for any function $f \geq 0, f \in C^2(D)$ we have

$$\begin{aligned}
&\frac{1}{|\Pi_\alpha|} \sum_{B\in\Pi_\alpha, B\subset D_N} f(x_B/N)\varphi_B \\
&\qquad \geq (2\sqrt{2\gamma}-\beta)\log N \left(\frac{1}{|\Pi_\alpha|} \sum_{B\in\Pi_\alpha, B\subset D_N} f(x_B/N) - \frac{K\|f\|_\infty}{|\Pi_\alpha|}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&P_N(E_{K,\beta,\alpha} \cap \Omega^+_{D_N}) \\
&\quad \leq \exp\left(-\frac{((2\sqrt{2\gamma}-\beta)\log N(\frac{1}{|\Pi_\alpha|}\sum_B f(x_B/N) - cN^{-4(1-\alpha)}))^2}{2\mathrm{var}_N(\frac{1}{|\Pi_\alpha|}\sum_B f(x_B/N)\varphi_B)}\right).
\end{aligned}$$

Applying Lemma B.4.1 and Lemma B.4.2 completes the proof. $\square$

4.6 Entropic repulsion

Proof of Proposition 1.3.5. Let $P^+_N(\cdot) := P_N(\cdot|\Omega^+_N)$. We use the notations of section 4.4, in particular the box-structure, and first assume $x = 0$. Set $\overline{\varphi}_{\varepsilon N} := \overline{\varphi}_{\varepsilon N}(x)$. We claim that on the set $\{\overline{\varphi}_{\varepsilon N} \leq (2\sqrt{2\gamma} - \eta)\log N\} \cap \Omega^+_N$, there exists $\delta > 0$ such that

$$\sharp\{x \in V_{\varepsilon N} : \varphi_x \leq (2\sqrt{2\gamma} - \eta/2)\log N\} \geq \delta|V_{\varepsilon N}|.$$

If this was not the case, we would have

$$(1-\delta)(2\sqrt{2\gamma}-\eta/2)\log N \leq \overline{\varphi}_{\varepsilon N} \leq (2\sqrt{2\gamma}-\eta)\log N,$$

which is impossible if δ is small enough such that $(1-\delta)(2\sqrt{2\gamma}-\eta/2) > (2\sqrt{2\gamma}-\eta)$. Thus, if $\alpha \in (0,1)$, there exists a shift of the $N^\alpha-$sublattice Π_α such that for this particular shift

$$\begin{aligned}
&P_N^+\left(\sharp\{x \in V_{\varepsilon N} : \varphi_x \leq (2\sqrt{2\gamma}-\eta/2)\log N\} \geq \delta |V_{\varepsilon N}|\right) \\
&= P_N^+\left(\frac{1}{|V_{\varepsilon N}|}\sum_{x\in V_{\varepsilon N}} 1_{\{\varphi_x \leq (2\sqrt{2\gamma}-\eta/2)\log N\}} \geq \delta\right) \\
&\leq P_N^+\left(\frac{1}{|\{B\in\Pi_\alpha, x_B \in V_{\varepsilon N}\}|}\sum_{B\in\Pi_\alpha, x_B\in V_{\varepsilon N}} 1_{\{\varphi_{x_B} \leq (2\sqrt{2\gamma}-\eta/2)\log N\}} \geq \delta\right).
\end{aligned}$$

(This is true since $\frac{1}{|V_{\varepsilon N}|}\sum_{x\in V_{\varepsilon N}} 1_{\{\varphi_x\leq(2\sqrt{2\gamma}-\eta/2)\log N\}}$ is the average over all possible such shifts of the $N^\alpha-$lattice). Let $S_\alpha := \{B \in \Pi_\alpha, x_B \in V_{\varepsilon N}\}$ for this particular Π_α. Choose $0 < \delta' < \delta$. Then

$$\begin{aligned}
P_N^+&\left(\frac{1}{|S_\alpha|}\sum_{B\in S_\alpha} 1_{\{\varphi_{x_B}\leq(2\sqrt{2\gamma}-\eta/2)\log N\}} \geq \delta\right) \\
&\leq P_N^+\left(\frac{1}{|S_\alpha|}\sum_{B\in S_\alpha} 1_{\{\varphi_B\leq(2\sqrt{2\gamma}-\eta/4)\log N\}} \geq \delta'\right) \\
&\quad + P_N^+\left(\frac{1}{|S_\alpha|}\sum_{B\in S_\alpha} 1_{\{\varphi_B-\varphi_{x_B}>(\eta/4)\log N\}} \geq (\delta-\delta')\right).
\end{aligned} \tag{4.44}$$

We have $|S_\alpha| \geq c\varepsilon N^{4(1-\alpha)}$. Thus

$$\begin{aligned}
P_N^+&\left(\frac{1}{|S_\alpha|}\sum_{B\in S_\alpha} 1_{\{\varphi_B\leq(2\sqrt{2\gamma}-\eta/4)\log N\}} \geq \delta'\right) \\
&\leq P_N^+\left(\sharp\{B\in\Pi_\alpha : \varphi_B \leq (2\sqrt{2\gamma}-\eta/4)\log N)\} \geq c\delta'\varepsilon N^{4(1-\alpha)}\right).
\end{aligned}$$

But in the proof of the upper bound of Theorem 1.3.4 we have seen that

$$P_N(E^c_{k,\beta,\alpha}\cap\Omega_N^+) \leq \exp\left(-(8\gamma\mathcal{C}_V^2(D)+1)(\log N)^2\right),$$

hence for large enough N,

$$P_N^+\big(\sharp\{B \in \Pi_\alpha : \varphi_B \le (2\sqrt{2\gamma} - \eta/4)\log N\} \ge c\delta'\varepsilon N^{4(1-\alpha)}\big)$$
$$\le \exp\left(-c(\log N)^2\right).$$

Thus what is left is the second term in (4.44). Note

$$P_N\big(\varphi_B - \varphi_{x_B} > (\eta/4)\log N | \mathcal{F}_\alpha\big) \le \exp(-c\eta^2 \log N).$$

Let $\theta_B : 1_{\{\varphi_B - \varphi_{x_B} > (\eta/4)\log N\}}$. As in the proof of Theorem 1.3.3 we have, using Lemma 11 of [3], for large N,

$$P_N\bigg(\sum_{B\in S_\alpha} 1_{\{\varphi_B - \varphi_{x_B} > (\eta/4)\log N\}} \ge (\delta - \delta')|S_\alpha|\bigg)$$
$$\le P_N\bigg(\big|\sum_{B\in S_\alpha}(\theta_B - E\theta_B)\big| \ge c\varepsilon N^{4(1-\alpha)}\big((\delta - \delta') - N^{-c'\eta^2}\big)\bigg)$$
$$\le P_N\bigg(\big|\sum_{B\in S_\alpha}(\theta_B - E\theta_B)\big| \ge c\varepsilon(\delta - \delta')N^{4(1-\alpha)}\bigg)$$
$$\le 2\exp\Big(-c\varepsilon(\delta - \delta')N^{4(1-\alpha)}\Big).$$

Together with Theorem 1.3.4, this proves

$$\lim_{N\to\infty} P_N(\overline{\varphi}_{\varepsilon N} \le (2\sqrt{2\gamma} - \eta)\log N \,|\, \Omega_N^+) = 0$$

if $x = 0$. For arbitrary x repeat the argument with a shifted grid. $\square$

Appendix A
Discrete Sobolev Norms

A.1 Norm estimates

In this section, we prove some basic estimates on the discrete Sobolev norms which are used in the proof of the regularity for the solution of the Dirichlet problem. Recall

$$E_1 = \{v : V_N \cup \partial_2 V_N \to \mathbb{R} : v(x) = 0 \ \forall x \in \partial_2 V_N\}$$

and for $v, w \in E_1$,

$$\mathcal{D}(v, w) := \sum_{x \in V_N} \Delta v(x) \Delta w(x) + \sum_{x \in \partial_- V_N} r(x) v(x) w(x).$$

Lemma A.1.1 *Let $v \in E_1$. There exists a constant c depending on the dimension such that*

$$\sum_{x \in V_N} \sum_{i=1}^{d} \sum_{j=1}^{d} (\nabla_i \nabla_j v(x))^2 \leq c\mathcal{D}(v, v).$$

Proof Expanding the square gives

$$\begin{aligned}
&(2d)^2 \sum_{x \in V_N} (\Delta v(x))^2 \\
&\quad = \sum_{x \in V_N} \sum_{i,j=1}^{d} \big(4v(x)^2 - 2v(x)v(x+e_i) - 2v(x)v(x-e_i) \\
&\qquad - 2v(x)v(x+e_j) - 2v(x)v(x-e_j) + v(x+e_i)v(x+e_j) \\
&\qquad + v(x+e_i)v(x-e_j) + v(x-e_i)v(x+e_j) + v(x-e_i)v(x-e_j)\big).
\end{aligned} \tag{A.45}$$

Now, taking the geometry of V_N and the 0-boundary conditions outside V_N into consideration, we can shift the summation, and obtain for any e_i with $|e_i| = 1$,

$$\sum_{x\in V_N} v(x)^2 = \sum_{x\in V_N} v(x+e_i)^2 + \sum_{\substack{x\in\partial_2 V_N:\\ x+e_i\in V_N}} v(x+e_i)^2.$$

Similarly, we have

$$\begin{aligned}\sum_{x\in V_N} v(x)v(x-e_i) &= \sum_{x\in V_N} v(x+e_i)v(x)\\ &= \sum_{x\in V_N} v(x+e_i+e_j)v(x+e_j)\\ &\quad+ \sum_{\substack{x\in\partial_2 V_N:\\ x+e_i+e_j\in V_N\\ x+e_j\in V_N}} v(x+e_i+e_j)v(x+e_j),\end{aligned}$$

and

$$\sum_{x\in V_N} v(x-e_i)v(x+e_j) = \sum_{x\in V_N} v(x+e_i+e_j)v(x).$$

Furthermore, if $i \neq j$

$$\sum_{x\in V_N} v(x-e_i)v(x-e_j) = \sum_{x\in V_N} v(x+e_i)v(x+e_j)$$

and

$$\sum_{x\in V_N} v(x-e_i)^2 = \sum_{x\in V_N} v(x+e_i)^2 - \sum_{\substack{x\in\partial V_N:\\ x-e_i\in V_N}} v(x-e_i)^2 + \sum_{\substack{x\in\partial V_N:\\ x+e_i\in V_N}} v(x+e_i)^2.$$

We define the following quantities.

$$T_1 := \sum_{i,j=1}^{d} \sum_{\substack{x\in\partial_2 V_N:\\ x+e_i\in V_N\\ x+e_j\in V_N\\ x+e_i+e_J\in V_N}} \left(v(x+e_i)+v(x+e_j)-v(x+e_i+e_j)\right)^2,$$

$$T_2 := \sum_{i,j=1}^{d} \sum_{\substack{x\in\partial_2 V_N:\\ x+e_i\in V_N\\ x+e_j\notin V_N\\ x+e_i+e_J\in V_N}} \left(v(x+e_i) - v(x+e_i+e_j)\right)^2,$$

$$T_3 := \sum_{i,j=1}^{d} \sum_{\substack{x\in\partial_2 V_N:\\ x+e_i\notin V_N\\ x+e_j\in V_N\\ x+e_i+e_J\in V_N}} \left(v(x+e_j) - v(x+e_i+e_j)\right)^2,$$

$$T_4 := \sum_{i,j=1}^{d} \Big[\sum_{\substack{x\in\partial_2 V_N:\\ x+e_i\in V_N\\ x+e_j\notin V_N\\ x+e_i+e_J\notin V_N}} v(x+e_i)^2 + \sum_{\substack{x\in\partial_2 V_N:\\ x+e_i\notin V_N\\ x+e_j\in V_N\\ x+e_i+e_J\notin V_N}} v(x+e_j)^2 + \sum_{\substack{x\in\partial_2 V_N:\\ x+e_i\notin V_N\\ x+e_j\notin V_N\\ x+e_i+e_J\in V_N}} v(x+e_i+e_j)^2 \Big]$$

and

$$T_5 := \sum_{\substack{x\in\partial V_N:\\ x-e_i\in V_N}} v(x-e_i)^2$$

Note that T_1 to T_5 are nonnegative, and $T_5 \le \sum_{x\in\partial_- V_N} r(x)v(x)^2$. By the above considerations, the right-hand side of (A.45) can be rewritten and bounded as follows

$$\begin{aligned}
&(2d)^2 \sum_{x\in V_N} \left(\Delta v(x)\right)^2 \\
&= \sum_{x\in V_N} \sum_{i,j=1}^{d} \Big(v(x)^2 + v(x+e_i)^2 + v(x+e_j)^2 + v(x+e_i+e_j)^2 \\
&\quad - 2v(x)v(x+e_i) - 2v(x+e_i+e_j)v(x+e_j) - 2v(x)v(x+e_j) \\
&\quad - 2v(x+e_i+e_j)v(x+e_i) + v(x+e_i)v(x+e_j) + 2v(x+e_i+e_j)v(x) \\
&\quad + v(x+e_i)v(x+e_j)\Big) + T_1 + T_2 + T_3 + T_4 - T_5 \\
&\ge \sum_{i,j=1}^{d} \sum_{x\in V_N} (\nabla_i \nabla_j v(x))^2 - \sum_{x\in\partial_- V_N} r(x)v(x)^2.
\end{aligned}$$

Thus

$$\sum_{i,j=1}^{d} \sum_{x \in V_N} (\nabla_i \nabla_j v(x))^2 \le (2d)^2 \sum_{x \in V_N} (\Delta v(x))^2 + \sum_{x \in \partial_- V_N} r(x) v(x)^2$$
$$\le (2d)^2 \mathcal{D}(v,v),$$

which proves the lemma. □

Lemma A.1.2 *Let $v \in E_1$. There exists $c > 0$ such that*

$$\sum_{x \in V_N} v(x)^2 \le cN^2 \left(\sum_{x \in V_N} \sum_{i=1}^{d} (\nabla_i v(x))^2 + \sum_{x \in \partial_- V_N} r(x) v(x)^2 \right).$$

Proof Let $x \in V_N$ and denote $A_x^i := \{y \in V_N : \exists k \in \mathbb{Z} \text{ such that } y = x + k \cdot e_i\}$. Then

$$v(x)^2 = (v(x) - v(x+e_i) + v(x+e_i) - v(x+2e_i) + ... + v(x+k_0 e_i))^2,$$

where $k_0 \in \mathbb{N}$ such that $x + k_0 e_i \in \partial_- V_N$. Obviously $k_0 \le 2N$, thus using the fact that $(a+b)^2 \le 2a^2 + 2b^2$ for real numbers a, b we get

$$v(x)^2 \le 2N((v(x) - v(x+e_i))^2 + ...$$
$$... + (v(x+(k_0-1)e_i) - v(x+k_0 e_i))^2 + v(x+k_0 e_i)^2).$$

In the same way, we obtain

$$v(x)^2 \le 2N((v(x) - v(x-e_i))^2 + ...$$
$$... + (v(x-(k_1-1)e_i) - v(x-k_1 e_i))^2 + v(x+k_1 e_i)^2)$$

for some $k_1 \le 2N$, with $x - k_1 e_1 \in \partial_- V_N$. This gives

$$\sum_{x \in V_N} v(x)^2 \le 2 \sum_{x \in V_N} N \left(\sum_{y \in A_x^i} (v(y) - v(y+e_i))^2 + \sum_{y \in \partial_- V_N \cap A_x^i} v(y)^2 \right)$$
$$\le cN^2 \left(\sum_{x \in V_N} (v(x) - v(x+e_i))^2 + \sum_{x \in \partial_- V_N} r(x) v(x)^2 \right).$$

Since this inequality holds for any $1 \le i \le d$, the lemma is proven. □

Lemma A.1.3 *Let $v \in E_1$. There exists $c > 0$ such that for all $1 \le i \le d$*

$$\sum_{x\in V_N} (v(x+e_i)-v(x))^2 \le cN^2 \left(\sum_{x\in V_N} (\nabla_i \nabla_i v(x))^2 + \sum_{x\in\partial_- V_N} r(x)v(x)^2 \right).$$

Proof Let $h(x) := \nabla_i v(x)$ and repeat the arguments of the proof of Lemma A.1.2. □

From Lemmas A.1.2 and A.1.3 the following is clear:

Corollary A.1.4 *Let $v \in E_1$. There exists $c > 0$ such that*

$$\|v\|^2_{H^2(V_N)} \le cN^4 \left(\sum_{x\in V_N} \sum_{i,j=1}^{d} (\nabla_i \nabla_j v(x))^2 + \sum_{x\in\partial_- V_N} r(x)v(x)^2 \right).$$

Remark A.1.5 *Iterating this procedure, one evidently obtains for any $v : V_N \cup \partial_k V_N \to \mathbb{R}$ such that $v(x) = 0$ for $x \in \partial_k V_N$, that*

$$\|v\|^2_{H^k(V_N)} \le cN^{2k} \left(\sum_{x\in V_N} \sum_{\alpha:|\alpha|=k} (\nabla^\alpha v(x))^2 + \sum_{x\in\partial_- V_N} r(x)v(x)^2 \right).$$

Corollary A.1.6 *Let $v \in E_1$. There is $c > 0$ such that*

$$\|v\|^2_{H^2(V_N)} \le cN^4 \mathcal{D}(v,v).$$

Proof Follows from Lemma A.1.1 and Corollary A.1.4. □

Remark A.1.7 *This also proves that $\mathcal{D}(\cdot,\cdot)$ is positive definite.*

Corollary A.1.8 *Let $v \in E_1$ additionally fullfil $v(x) = 0$ for all $x \in \partial_- V_N$. Then there is $c > 0$ such that*

$$\|v\|^2_{H^2(V_N)} \ge cN^4 \mathcal{D}(v,v).$$

Proof Clear from the proof of Lemma A.1.1. □

A.2 Discrete Sobolev imbedding

The following reults are the discrete analogues of the Sobolev Imbedding Theorems. For completeness, we include the proofs of the versions we use.

Proposition A.2.1 *Let $f : \mathbb{Z}^d \to \mathbb{R}$ such that $f(x) = 0$ on V_N^c, and $\|f\|_{H^k(V_N)} \leq cN^{d/2}$ for some constant c independent of N. If $k > d/2$, then there exists $C > 0$ independent of N such that* $\sup_{x\in V_N} |f(x)| < C$*.*

Proof Let $\widehat{f}(t) = \sum_{x\in\mathbb{Z}^d} f(x)e^{i<t,x>}$ denote the Fourier transform of a function $f : \mathbb{Z}^d \to \mathbb{R}$. Then we have

$$\begin{aligned}
\widehat{\nabla_k f}(t) =& \sum_{x\in\mathbb{Z}^d} (f(x+e_k) - f(x))e^{i<t,x>} \\
=& \sum_{x\in\mathbb{Z}^d} (f(x)e^{i<t,x-e_k>} - f(x)e^{i<t,x>}) \\
=& \widehat{f}(t)(e^{-it_k} - 1).
\end{aligned}$$

Iterating, we obtain

$$\widehat{\nabla_{k_1}...\nabla_{k_l}f}(t) = \widehat{f}(t)(e^{-it_{k_1}} - 1)...(e^{-it_{k_l}} - 1). \tag{A.46}$$

Fix $t_0 < 1$.Set $\mathbb{T}_d := [-\pi, \pi]^d$ and $A := [-t_0, t_0]^d$. Using the inverse Fourier transform we have

$$\begin{aligned}
f(x) =& c\int_{\mathbb{T}_d} \widehat{f}(t)e^{-i<t,x>}dt \\
=& c\int_A \widehat{f}(t)e^{-i<t,x>}dt + c\int_{\mathbb{T}_d\setminus A} \widehat{f}(t)e^{-i<t,x>}dt
\end{aligned}$$

For the second integral, note that on $\mathbb{T}_d \setminus A$ we have $|(e^{-it_k} - 1)| \geq |(e^{-it_0} - 1)| \geq c(t_0)$ for some positive constant $c(t_0)$, which by the

Plancherel Theorem implies for $l \in \mathbb{N}$

$$
\begin{aligned}
&|\int\limits_{\mathbb{T}_d \setminus A} \widehat{f}(t) e^{-i<t,x>} dt| \\
&\quad =|\int\limits_{\mathbb{T}_d \setminus A} (\widehat{\nabla_{k_1}...\nabla_{k_l}} f)(t)((e^{-it_{k_1}}-1)...(e^{-it_{k_l}}-1))^{-1} e^{-i<t,x>} dt| \\
&\quad \leq c N^{-l} \|f\|_{H^l(V_N)}
\end{aligned}
$$

which is bounded by assumption if $l > d/2$. The first integral we can treat as follows: Note that by (A.46), using Taylor expansion, we have for any $j \in \mathbb{N}$,

$$
|\widehat{f}(t)|^2 \cdot |t|^{2j} \leq c \cdot |\widehat{f}(t)|^2 |(e^{-it_{k_1}}-1) \cdot ... \cdot (e^{-it_{k_l}}-1)|^2 \leq |\widehat{\nabla_{k_1}...\nabla_{k_l}} f(t)|^2.
$$

This yields

$$
\begin{aligned}
\int\limits_A |\widehat{f}(t)| dt = &\int\limits_A \frac{1}{(1+N^2|t|^2)^{l/2}} (1+N^2|t|2)^{l/2} |\widehat{f}(t)| dt \\
&\leq \left(\int\limits_A \frac{1}{(1+N^2|t|^2)^l} dt \right)^{1/2} \cdot \left(\int\limits_A (1+N^2|t|^2)^l |\widehat{f}(t)|^2 dt \right)^{1/2} \\
&\leq c N^{-l} \cdot \left(\int\limits_A \sum_{j=0}^{l} (N|t|)^{2j} |\widehat{f}(t)|^2 dt \right)^{1/2} \\
&\leq c N^{-l} \|f\|_{H^l(V_N)} \leq c N^{d/2-l},
\end{aligned}
$$

using the Plancherel Theorem once more. □

This implies

Corollary A.2.2 *Let $f : \mathbb{Z}^d \to \mathbb{R}$ such that $f(x) = 0$ on V_N^c, and $\|f\|_{H^k(V_N)} \leq c N^{d/2}$ for some constant c independent of N. If $k > d/2 + l$, then there exists $C > 0$ independent of N such that $\sup_{x \in V_N} |\nabla^\alpha f(x)| \leq \frac{C}{N^{|\alpha|}}$ for all $0 \leq |\alpha| \leq l$.*

Appendix B
$k-$capacity

B.3 $d \geq 5$

In this section, we prove the equivalence of the several expressions for the "capacity"$\mathcal{C}_k(V)$. They are obtained from the decay of the Green's function (Lemma 1.2.6). Define a function $g_k : \mathbb{R}^d \to \mathbb{R}$ by $g_k(x) = \frac{\eta_k}{q_k}|x|^{2k-d}$ and a positive compact operator K_k on $L^2(V)$ by

$$K_k f(x) = \int_V g_k(x-y)f(y)dy \quad (x \in V)\,.$$

From the above lemma we get, for $|x-y| \to \infty$,

$$|g_k(x-y) - G(x,y)| = o(|x-y|^{-d+2k})\,. \tag{B.47}$$

In this section we use the short notation $\langle f, g\rangle_V := \int_V f(x)g(x)dx$, for suitable functions f, g and $V \subset \mathbb{R}^d$. Note first the following (see also [20], Lemma 5.2):

Lemma B.3.1 *Let $d \geq 2k+1$. For all $h, f \in H^k(V)$,*

$$\frac{q_k}{(2d)^k}\langle h, K_k \frac{1}{(2d)^k}(-\Delta)^k f\rangle_V = \langle h, f\rangle_V.$$

Proof In order to distinguish between the discrete and the continuous Laplacian, we denote them by Δ_d and Δ_c respectively. Using (B.47) we obtain

$$\begin{aligned}
&\frac{q_k}{(2d)^k}\langle h, K_k(-\Delta_c)^k f\rangle_V \\
&= \lim_{N\to\infty}\frac{1}{N^{2d}}\sum_{x\in V_N}\sum_{y\in V_N} h\left(\frac{x}{N}\right) g_k\left(\frac{x}{N}-\frac{y}{N}\right)\frac{q_k}{(2d)^k}\left((-\Delta_c)^k f\right)\left(\frac{y}{N}\right) \\
&= \lim_{N\to\infty}\frac{1}{N^{2d}}\sum_{x\in V_N} h\left(\frac{x}{N}\right)\sum_{y\in V_N}\Big[N^{d-2k}G\left(\frac{x}{N},\frac{y}{N}\right) \\
&\qquad\qquad \times\frac{1}{N^{-2k}}q_k\left((-\Delta_d)^k f\right)\left(\frac{y}{N}\right)\Big] \\
&= \lim_{N\to\infty}\frac{1}{N^{d}}\sum_{x\in V_N} h\left(\frac{x}{N}\right)\cdot f\left(\frac{x}{N}\right) = \frac{1}{q_k}\langle h, f\rangle_V.
\end{aligned}$$

□

We can now prove the equivalence of several expressions for the $k-$th order capacity $\mathcal{C}_{||}(\mathcal{V})$. Proposition B.3.2 below was used implicitly in Section 5 of [20] (Lemma 5.2). As we are not aware of a reference, we include the proof here.

Proposition B.3.2 *Let* $V=[-1,1]^d, d\geq 2k+1$. *Then*

$$\begin{aligned}
&\inf\left\{\frac{q_k}{(2d)^k}\int_{\mathbb{R}^d}|(-\nabla)^k h|^2dx : H\in H_0^k(\mathbb{R}^d), h\geq 1_V\right\} \\
&\quad=\sup\left\{2\langle f,1_V\rangle_V-\langle f,K_kf\rangle_V : f\in L_2(V)\right\} \\
&\quad=\sup\left\{\frac{\langle f,1_V\rangle_V^2}{\langle f,K_kf\rangle_V} : f\in L_2(V)\right\}.
\end{aligned}$$

Proof Let us prove the first equality. Note that $M:=\{h\in H_0^k(\mathbb{R}^d): h\geq 1_V\}$ is a closed convex subset of the Hilbert space $H_0^k(\mathbb{R}^d)$, and thus has a minimizer h_0 for the Sobolev-norm on $H_0^k(\mathbb{R}^d)$. But this means exactly that h_0 minimizes $\int_{\mathbb{R}^d}|(-\nabla)^k h|^2dx$ for $h\in M$. It is immediate that $h_0=1$ on V. Furthermore, $(-\Delta)^k h_0=0$ outside V. To see this, set $g(\varepsilon)=\int_{\mathbb{R}^d}|(-\nabla)^k h+\varepsilon\varphi|^2dx$ for some $\varphi\in C_c^\infty(\mathbb{R}^d\setminus V)$. Then $\frac{dg}{d\varepsilon}\Big|_{\varepsilon=0}=0$, because h_0 is a minimizer of the integral. But this implies $\langle(-\Delta)^k h_0,\varphi\rangle_{\mathbb{R}^d\setminus V}=\langle(-\nabla)^k h_0,(-\nabla)^k\varphi\rangle_{\mathbb{R}^d\setminus V}=0$ for all $\varphi\in$

$C_c^\infty(\mathbb{R}^d \setminus V)$ and thus $(-\Delta)^k h_0 = 0$ on $\mathbb{R}^d \setminus V$.
There exist $\tau_n \in C_0^\infty(\mathbb{R}^d), n \in \mathbb{N}$ such that $\lim_{n\to\infty}\langle h_0 - \tau_n, (-\Delta)^k(h_0 - \tau_n)\rangle_{\overset{\circ}{V}} = 0$ and $\tau_n = h_0$ on $\mathbb{R}^d \setminus V$, where $\overset{\circ}{V}$ denotes the interior of V. Let

$$f_n = \frac{q_k}{(2d)^k}(-\Delta)^k \tau_n \,.$$

For every n, f_n belongs to $L_2(\mathbb{R}^d)$, and, by the fact that $f_n = 0$ outside V, Lemma B.3.1 and integration by parts yield

$$\begin{aligned} 2\langle f_n, \tau_n\rangle_V - \langle f_n, K_k f_n\rangle_V &= \frac{q_k}{(2d)^k}\langle(-\Delta)^k \tau_n, \tau_n\rangle_{\mathbb{R}^d} \\ &= \frac{q_k}{(2d)^k}\int\limits_{\mathbb{R}^d} |(-\nabla)^k \tau_n|^2 dx \,. \end{aligned}$$

Moreover, as in [2], $\lim_{n\to\infty} |< f_n, 1_V - \tau_n >_{L^2(\mathbb{R}^d)}| = 0$. Together with the above this yields

$$\begin{aligned} \sup\{2\langle f, 1_V\rangle_V - \langle f, K_k f\rangle_V\} &\geq \limsup_{n\to\infty}\{2\langle f_n, 1_V\rangle_V - \langle f_n, K_k f_n\rangle_V\} \\ &= \frac{q_k}{(2d)^k}\int\limits_{\mathbb{R}^d} |(-\nabla)^k h_0|\, dx \,, \end{aligned}$$

which gives one direction in the first equation. The other direction is an elementary calculation based on Lemma B.3.1.
The second equation follows by expanding f in a basis of eigenvectors of the compact positive operator K_k. Maximising shows that both sides are equal to $\sum_{i\in\mathbb{N}} \frac{\langle e_i, 1_V\rangle^2}{\lambda_i}$, where the e_i are the eigenvectors of K_k and λ_i the corresponding eigenvalues. □

B.4 $d = 4$

In the finite volume case, the proof is very similar to the one in the last section. Let

$$E_0 := \{f : V_N \cup \partial_2 V_N \to \mathbb{R} : \|f\|_{H^k(V_N)} \leq cN^{d/2}, f(x) = 0\ \forall x \in \partial_- V_N\}.$$

If $f : V \to \mathbb{R}$, we write f_N for the function $V_N \to \mathbb{R}$, $f_N(x) := f(x/N)$.

Lemma B.4.1

$$\inf\{\|\Delta_N h\|^2_{L_2(V_N)} : \ h \in E_0, h \geq 1 \ on \ D_N\}$$
$$= \sup\left\{\langle 1_{D_N}, f_N\rangle_{D_N} - \frac{1}{2}\langle f_N G_N f_N\rangle : f \in L_2(V_N) : f = 0 \ on \ V_N \setminus D_N\right\}$$
$$= \sup\left\{\frac{\langle 1_{D_N}, f_N\rangle^2_{D_N}}{2\langle f_N, G_N f_N\rangle_{D_N}} : f \in L_2(V_N) : f = 0 \ on \ V_N \setminus D_N\right\}.$$

Proof We start with the first equality. Since $E_0(V_N)$ is finite dimensional, there exists a minimizer $h_N^{(0)}$. Obviously, $h_N^{(0)} = 1$ on D_N. Furthermore, $\Delta^2 h_N^{(0)} = 0$ outside D_N. To see this, set $\psi(\varepsilon) = \sum_{x \in V_N} |\Delta h_N^{(0)}(x) + \varepsilon\varphi(x)|$ for any test function $\varphi : V_N \cup \partial_2 V_N \to \mathbb{R}$, with $\varphi(x) = 0$ for all $x \in V_N \setminus D_N$. Then $\frac{d\psi}{d\varepsilon}\Big|_{\varepsilon=0} = 0$, because $h_N^{(0)}$ is a minimizer of the norm. But this implies $\langle \Delta^2 h_N^{(0)}, \varphi\rangle_{V_N} = \langle \Delta h_N^{(0)}, \Delta\varphi\rangle_{V_N} = 0$ for all φ as above, and thus the claim.
Set
$$f_N = \Delta_N^2 h_N^{(0)}.$$
By the fact that $f_N^{(n)} = 0$ outside D_N, summation by parts gives

$$2\langle f_N, h_N^{(0)}\rangle_{D_N} - \langle f_N, G_N f_N\rangle_{D_N} = \sum_{x \in V_N} |\Delta h_N^{(0)}|^2.$$

The above this yields

$$\sup\left\{\langle 1_{D_N}, f_N\rangle_{D_N} - \frac{1}{2}\langle f_N G_N f_N\rangle : f \in L_2(V_N) : f = 0 \text{ on } V_N \setminus D_N\right\}$$
$$\geq 2\langle f_N, h_N^{(0)}\rangle_{D_N} - \langle f_N, G_N f_N\rangle_{D_N}$$
$$= \sum_{x \in V_N} |\Delta_N h_N^{(0)}|^2,$$

which is one direction in the first equation. The other direction is an elementary calculation.
The second equation follows by expanding f in a basis of eigenvectors of the symmetric matrix G_N. Maximizing shows that both sides are equal to $\sum_{i \in \mathbb{N}} \frac{<e_i, 1_D>^2}{\lambda_i}$, where the e_i are the eigenvectors and λ_i the corresponding eigenvalues. $\square$

Lemma B.4.2 *With the above notations,*

$$\lim_{N\to\infty} \inf\{\|\Delta_N h\|^2_{L_2(V_N)} :\ h \in H_0^2, h \geq 1 \text{ on } D_N\} = \mathcal{C}_V^2(D).$$

Proof $\{h \in H_0^2(V) : h \geq 1_D\}$ is a closed convex subset of the Hilbert space $H_0^2(V)$, and therefore there exists a minimizer h_0 for $\int_V |\Delta h|^2 dx$. For every $n \in \mathbb{N}$, the discretisation $h_{0,N}(x) := h_0(x/N)$ belongs to $H_0^2(V_N)$, which proves one direction. Let $\varepsilon > 0$. For every $N \in \mathbb{N}$ we can find $\tilde{h}^{(N)} \in H_0^2(V)$ such that $\tilde{h}^{(N)} \geq 1_D$ and the discretisation $\tilde{h}_N^{(N)}$ of $\tilde{h}^{(N)}$ is equal to $h_N^{(0)}$ of the proof of Lemma B.4.1. If N is large enough, $\|\tilde{h}_N^{(N)}\|_{L_2(V_N)} \geq \|\tilde{h}^{(N)}\|_{L_2(V)} - \varepsilon$. Since h_0 is a minimizer, $\liminf_{N\to\infty} \|h_N^{(0)}\|_N \geq \liminf_{N\to\infty} \|\tilde{h}^{(N)}\|_{L_2(V)} - \varepsilon \geq \|h_0\|_{L_2(V)} - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the claim is proven. $\square$

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Dissertation

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